HL Paper 1

Consider the function $f : x \to \sqrt{\frac{\pi}{4} - \arccos x}$.

(a) Find the largest possible domain of *f*.

(b) Determine an expression for the inverse function, f^{-1}, and write down its domain.

Markscheme

(a) $\frac{\pi}{4} - \arccos x \geqslant 0$

$\arccos x \leqslant \frac{\pi}{4}$ ***(M1)***

$x \geqslant \frac{\sqrt{2}}{2}$ $\left(\text{accept } x \geqslant \frac{1}{\sqrt{2}}\right)$ ***(A1)***

since $-1 \leqslant x \leqslant 1$ ***(M1)***

$\Rightarrow \frac{\sqrt{2}}{2} \leqslant x \leqslant 1$ $\left(\text{accept } \frac{1}{\sqrt{2}} \leqslant x \leqslant 1\right)$ ***A1***

Note: Penalize the use of $<$ instead of $\leqslant$ only once.

(b) $y = \sqrt{\frac{\pi}{4} - \arccos x} \Rightarrow x = \cos\left(\frac{\pi}{4} - y^2\right)$ ***M1A1***

$f^{-1} : x \to \cos\left(\frac{\pi}{4} - x^2\right)$ ***A1***

$0 \leqslant x \leqslant \sqrt{\frac{\pi}{4}}$ ***A1***

[8 marks]

Examiners report

Very few correct solutions were seen to (a). Many candidates realised that $\arccos x \leqslant \frac{\pi}{4}$ but then concluded incorrectly, not realising that cos is a decreasing function, that $x \leqslant \cos\left(\frac{\pi}{4}\right)$. In (b) candidates often gave an incorrect domain.

A function f is defined by $f(x) = \frac{3x-2}{2x-1}$, $x \in \mathbb{R}$, $x \neq \frac{1}{2}$.

a. Find an expression for $f^{-1}(x)$. [4]

b. Given that $f(x)$ can be written in the form $f(x) = A + \frac{B}{2x-1}$, find the values of the constants A and B. [2]

c. Hence, write down $\int \frac{3x-2}{2x-1} \mathrm{d}x$. [1]

Markscheme

a. $f : x \to y = \frac{3x-2}{2x-1}$ $\quad f^{-1} : y \to x$

$y = \frac{3x-2}{2x-1} \Rightarrow 3x - 2 = 2xy - y$ ***M1***

$\Rightarrow 3x - 2xy = -y + 2$ ***M1***

$x(3 - 2y) = 2 - y$

$x = \frac{2-y}{3-2y}$ ***A1***

$\left(f^{-1}(y) = \frac{2-y}{3-2y}\right)$

$f^{-1}(x) = \frac{2-x}{3-2x} \quad \left(x \neq \frac{3}{2}\right)$ ***A1***

Note: x and y might be interchanged earlier.

Note: First ***M1*** is for interchange of variables second ***M1*** for manipulation

Note: Final answer must be a function of x

[4 marks]

b. $\frac{3x-2}{2x-1} = A + \frac{B}{2x-1} \Rightarrow 3x - 2 = A(2x - 1) + B$

equating coefficients $3 = 2A$ and $-2 = -A + B$ ***(M1)***

$A = \frac{3}{2}$ and $B = -\frac{1}{2}$ ***A1***

Note: Could also be done by division or substitution of values.

[2 marks]

c. $\int f(x)\mathrm{d}x = \frac{3}{2}x - \frac{1}{4}\ln|2x - 1| + c$ ***A1***

Note: accept equivalent e.g. $\ln|4x - 2|$

[1 mark]

Total [7 marks]

Examiners report

a. Well done. Only a few candidates confused inverse with derivative or reciprocal.

b. Not enough had the method of polynomial division.

c. Reasonable if they had an answer to (b) (follow through was given) usual mistakes with not allowing for the derivative of the bracket.

Write $\ln(x^2 - 1) - 2\ln(x + 1) + \ln(x^2 + x)$ as a single logarithm, in its simplest form.

Markscheme

$\ln(x^2 - 1) - \ln{(x + 1)}^2 + \ln x(x + 1)$ ***(A1)***

$= \ln \frac{x(x^2-1)(x+1)}{(x+1)^2}$ ***(M1)A1***

$= \ln \frac{x(x+1)(x-1)(x+1)}{(x+1)^2}$ ***(A1)***

$= \ln x(x-1) \quad \left(= \ln(x^2 - x)\right)$ ***A1***

[5 marks]

Examiners report

There were fewer correct solutions to this question than might be expected. A significant number of students managed to combine the terms to form one logarithm, but rather than factorising, then expanded the brackets, which left them unable to gain an answer in its simplest form.

Consider the equation $yx^2 + (y-1)x + (y-1) = 0$.

a. Find the set of values of y for which this equation has real roots. [4]

b. Hence determine the range of the function $f : x \to \frac{x+1}{x^2+x+1}$. [3]

c. Explain why f has no inverse. [1]

Markscheme

a. for the equation to have real roots

$(y-1)^2 - 4y(y-1) \geqslant 0$ ***M1***

$\Rightarrow 3y^2 - 2y - 1 \leqslant 0$

(by sign diagram, or algebraic method) ***M1***

$-\frac{1}{3} \leqslant y \leqslant 1$ ***A1A1***

Note: Award first ***A1*** for $-\frac{1}{3}$ and 1, and second ***A1*** for inequalities. These are independent marks.

[4 marks]

b. $f : x \to \frac{x+1}{x^2+x+1} \Rightarrow x + 1 = yx^2 + yx + y$ ***(M1)***

$\Rightarrow 0 = yx^2 + (y-1)x + (y-1)$ ***A1***

hence, from (a) range is $-\frac{1}{3} \leqslant y \leqslant 1$ ***A1***

[3 marks]

c. a value for y would lead to 2 values for x from (a) ***R1***

Note: Do not award ***R1*** if (b) has not been tackled.

[1 mark]

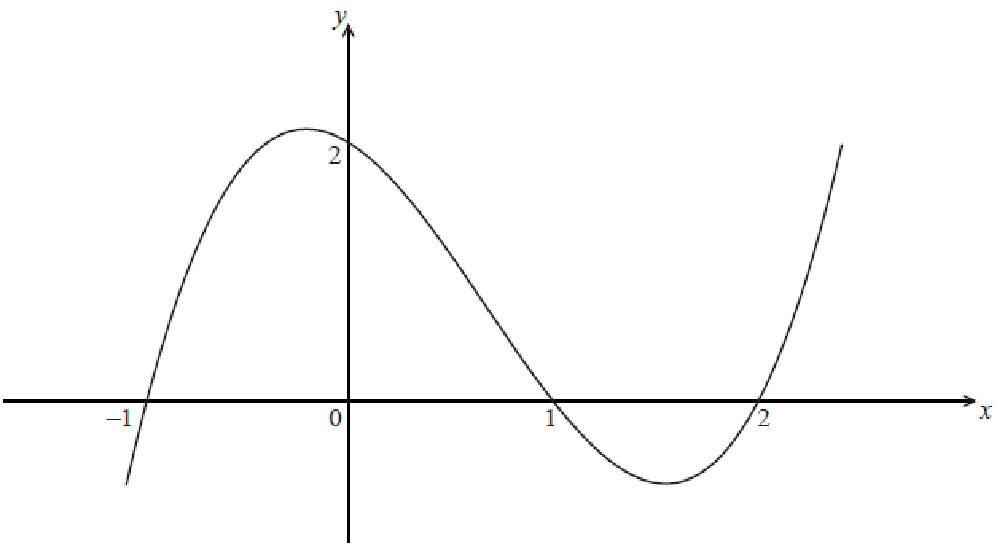
Examiners report

a. (a) The best answered part of the question. The critical points were usually found, but the inequalities were often incorrect. Few candidates were convincing regarding the connection between (a) and (b). This had consequences for (c).

b. (a) The best answered part of the question. The critical points were usually found, but the inequalities were often incorrect. Few candidates were convincing regarding the connection between (a) and (b). This had consequences for (c).

c. (a) The best answered part of the question. The critical points were usually found, but the inequalities were often incorrect. Few candidates were convincing regarding the connection between (a) and (b). This had consequences for (c).

Let $f(x) = x^3 + ax^2 + bx + c$, where a , b , $c \in \mathbb{Z}$. The diagram shows the graph of $y = f(x)$.



a. Using the information shown in the diagram, find the values of a , b and c . [4]

b. If $g(x) = 3f(x - 2)$, [3]

(i) state the coordinates of the points where the graph of g intercepts the x-axis.
(ii) Find the y-intercept of the graph of g .

Markscheme

a. **METHOD 1**

$f(x) = (x + 1)(x - 1)(x - 2)$ ***M1***
$= x^3 - 2x^2 - x + 2$ ***A1A1A1***
$a = -2$, $b = -1$ and $c = 2$
METHOD 2
from the graph or using $f(0) = 2$
$c = 2$ ***A1***
setting up linear equations using $f(1) = 0$ and $f(-1) = 0$ (or $f(2) = 0$) ***M1***
obtain $a = -2$, $b = -1$ ***A1A1***
[4 marks]

b. (i) $(1, 0)$, $(3, 0)$ and $(4, 0)$ ***A1***

(ii) $g(0)$ occurs at $3f(-2)$ ***(M1)***
$= -36$ ***A1***
[3 marks]

Examiners report

a. This question was well answered in general. Part b(ii) was often the most problematic, usually because of candidates going to the trouble of finding an explicit and sometimes incorrect expression for $f(x-2)$.

b. This question was well answered in general. Part b(ii) was often the most problematic, usually because of candidates going to the trouble of finding an explicit and sometimes incorrect expression for $f(x-2)$.

The function f is defined by $f(x) = \frac{3x}{x-2},\ x \in \mathbb{R},\ x \neq 2$.

a. Sketch the graph of $y = f(x)$, indicating clearly any asymptotes and points of intersection with the x and y axes. [4]

b. Find an expression for $f^{-1}(x)$. [4]

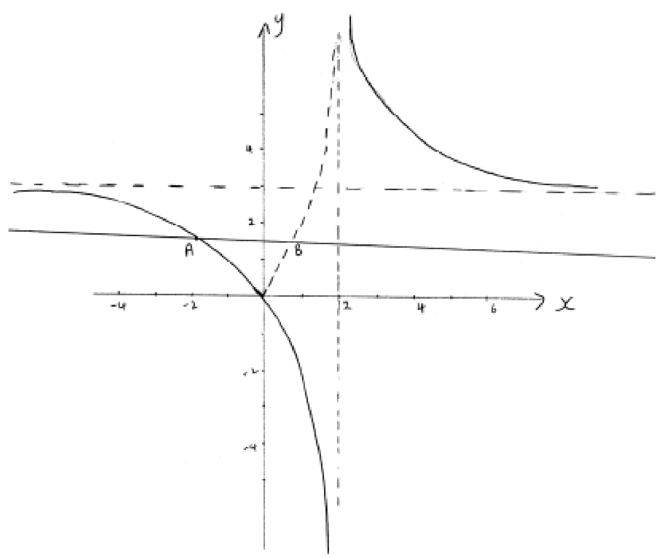
c. Find all values of x for which $f(x) = f^{-1}(x)$. [3]

d. Solve the inequality $|f(x)| < \frac{3}{2}$. [4]

e. Solve the inequality $f\left(|x|\right) < \frac{3}{2}$. [2]

Markscheme

a.



Note: In the diagram, points marked A and B refer to part (d) and do not need to be seen in part (a).

shape of curve ***A1***

Note: This mark can only be awarded if there appear to be both horizontal and vertical asymptotes.

intersection at $(0,\ 0)$ ***A1***

horizontal asymptote at $y = 3$ ***A1***

vertical asymptote at $x = 2$ ***A1***

[4 marks]

b. $y = \frac{3x}{x-2}$

$xy - 2y = 3x$ ***M1A1***

$xy - 3x = 2y$

$x = \frac{2y}{y-3}$

$\left(f^{-1}(x)\right) = \frac{2x}{x-3}$ ***M1A1***

Note: Final M1 is for interchanging of x and y, which may be seen at any stage.

[4 marks]

c. **METHOD 1**

attempt to solve $\frac{2x}{x-3} = \frac{3x}{x-2}$ ***(M1)***

$2x(x-2) = 3x(x-3)$

$x\left[2(x-2) - 3(x-3)\right] = 0$

$x(5-x) = 0$

$x = 0$ or $x = 5$ ***A1A1***

METHOD 2

$x = \frac{3x}{x-2}$ or $x = \frac{2x}{x-3}$ ***(M1)***

$x = 0$ or $x = 5$ ***A1A1***

[3 marks]

d. **METHOD 1**

at A : $\frac{3x}{x-2} = \frac{3}{2}$ AND at B : $\frac{3x}{x-2} = -\frac{3}{2}$ ***M1***

$6x = 3x - 6$

$x = -2$ ***A1***

$6x = 6 - 3x$

$x = \frac{2}{3}$ ***A1***

solution is $-2 < x < \frac{2}{3}$ ***A1***

METHOD 2

$\left(\frac{3x}{x-2}\right)^2 < \left(\frac{3}{2}\right)^2$ ***M1***

$9x^2 < \frac{9}{4}(x-2)^2$

$3x^2 + 4x - 4 < 0$

$(3x-2)(x+2) < 0$

$x = -2$ ***(A1)***

$x = \frac{2}{3}$ ***(A1)***

solution is $-2 < x < \frac{2}{3}$ ***A1***

[4 marks]

e. $-2 < x < 2$ ***A1A1***

Note: ***A1*** for correct end points, ***A1*** for correct inequalities.

Note: If working is shown, then ***A*** marks may only be awarded following correct working.

[2 marks]

Total [17 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]

A function is defined by $h(x) = 2\mathrm{e}^x - \frac{1}{\mathrm{e}^x},\ x \in \mathbb{R}$. Find an expression for $h^{-1}(x)$.

Markscheme

$x = 2\mathrm{e}^y - \frac{1}{\mathrm{e}^y}$ ***M1***

Note: The ***M1*** is for switching the variables and may be awarded at any stage in the process and is awarded independently. Further marks do not rely on this mark being gained.

$x\mathrm{e}^y = 2\mathrm{e}^{2y} - 1$

$2\mathrm{e}^{2y} - x\mathrm{e}^y - 1 = 0$ ***A1***

$\mathrm{e}^y = \frac{x \pm \sqrt{x^2+8}}{4}$ ***M1A1***

$y = \ln\left(\frac{x \pm \sqrt{x^2+8}}{4}\right)$

therefore $h^{-1}(x) = \ln\left(\frac{x + \sqrt{x^2+8}}{4}\right)$ ***A1***

since ln is undefined for the second solution ***R1***

Note: Accept $y = \ln\left(\frac{x + \sqrt{x^2+8}}{4}\right)$.

Note: The ***R1*** may be gained by an appropriate comment earlier.

[6 marks]

Examiners report

A significant number of candidates did not recognise the need for the quadratic formula in order to find the inverse. Even when they did most candidates who got this far did not recognise the need to limit the solution to the positive only. This question was done well by a very limited number of candidates.

The polynomial $P(x) = x^3 + ax^2 + bx + 2$ is divisible by $(x + 1)$ and by $(x - 2)$.

Find the value of a and of b, where $a,\ b \in \mathbb{R}$.

Markscheme

METHOD 1

As $(x + 1)$ is a factor of $P(x)$, then $P(-1) = 0$ ***(M1)***

$\Rightarrow a - b + 1 = 0$ (or equivalent) ***A1***

As $(x - 2)$ is a factor of $P(x)$, then $P(2) = 0$ ***(M1)***

$\Rightarrow 4a + 2b + 10 = 0$ (or equivalent) ***A1***

Attempting to solve for a and b ***M1***

$a = -2$ and $b = -1$ ***A1 N1***

[6 marks]

METHOD 2

By inspection third factor must be $x - 1$. ***(M1)A1***

$(x + 1)(x - 2)(x - 1) = x^3 - 2x^2 - x + 2$ ***(M1)A1***

Equating coefficients $a = -2$, $b = -1$ ***(M1)A1 N1***

[6 marks]

METHOD 3

Considering $\frac{P(x)}{x^2-x-2}$ or equivalent ***(M1)***

$\frac{P(x)}{x^2-x-2} = (x + a + 1) + \frac{(a+b+3)x+2(a+2)}{x^2-x-2}$ ***A1A1***

Recognising that $(a + b + 3)x + 2(a + 2) = 0$ ***(M1)***

Attempting to solve for a and b ***M1***

$a = -2$ and $b = -1$ ***A1 N1***

[6 marks]

Examiners report

Most candidates successfully answered this question. The majority used the factor theorem, but a few employed polynomial division or a method based on inspection to determine the third linear factor.

Consider the function f defined by $f(x) = x^2 - a^2,\ x \in \mathbb{R}$ where a is a positive constant.

The function g is defined by $g(x) = x\sqrt{f(x)}$ for $|x| > a$.

a.i. Showing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes. [2]

$y = f(x)$;

a.ii. Showing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes. [4]

$y = \frac{1}{f(x)}$;

a.iii Showing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes. [2]

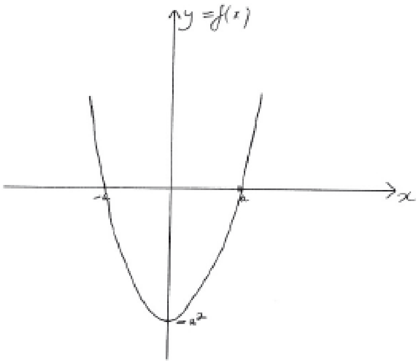
$y = \left|\frac{1}{f(x)}\right|$.

b. Find $\int f(x)\cos x\mathrm{d}x$. [5]

c. By finding $g'(x)$ explain why g is an increasing function. [4]

Markscheme

a.i.

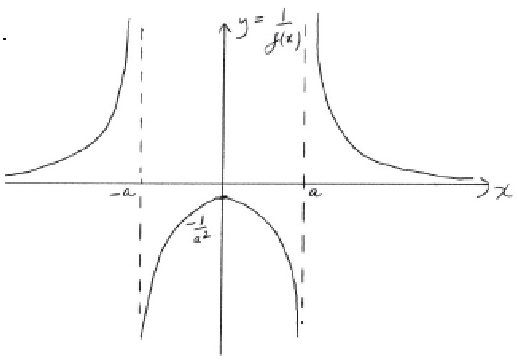


A1 for correct shape

A1 for correct x and y intercepts and minimum point

[2 marks]

a.ii.



A1 for correct shape

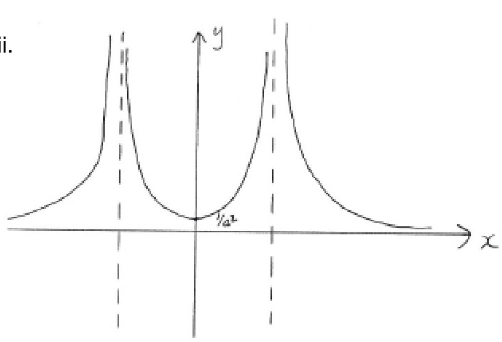
A1 for correct vertical asymptotes

A1 for correct implied horizontal asymptote

A1 for correct maximum point

[??? marks]

a.iii.



A1 for reflecting negative branch from (ii) in the x-axis

A1 for correctly labelled minimum point

[2 marks]

b. **EITHER**

attempt at integration by parts ***(M1)***

$\int (x^2 - a^2)\cos x\mathrm{d}x = (x^2 - a^2)\sin x - \int 2x\sin x\mathrm{d}x$ ***A1A1***

$= (x^2 - a^2)\sin x - 2\left[-x\cos x + \int \cos x\mathrm{d}x\right]$ ***A1***

$= (x^2 - a^2)\sin x + 2x\cos -2\sin x + c$ ***A1***

OR

$\int (x^2 - a^2)\cos x\mathrm{d}x = \int x^2\cos x\mathrm{d}x - \int a^2\cos x\mathrm{d}x$

attempt at integration by parts ***(M1)***

$\int x^2\cos x\mathrm{d}x = x^2\sin x - \int 2x\sin x\mathrm{d}x$ ***A1A1***

$= x^2\sin x - 2\left[-x\cos x + \int \cos x\mathrm{d}x\right]$ ***A1***

$= x^2\sin x + 2x\cos x - 2\sin x$

$-\int a^2\cos x\mathrm{d}x = -a^2\sin x$

$\int (x^2 - a^2)\cos x\mathrm{d}x = (x^2 - a^2)\sin x + 2x\cos x - 2\sin x + c$ ***A1***

[5 marks]

c. $g(x) = x(x^2 - a^2)^{\frac{1}{2}}$

$g'(x) = (x^2 - a^2)^{\frac{1}{2}} + \frac{1}{2}x(x^2 - a^2)^{-\frac{1}{2}}(2x)$ ***M1A1A1***

Note: Method mark is for differentiating the product. Award ***A1*** for each correct term.

$g'(x) = (x^2 - a^2)^{\frac{1}{2}} + x^2(x^2 - a^2)^{-\frac{1}{2}}$

both parts of the expression are positive hence $g'(x)$ is positive ***R1***

and therefore g is an increasing function (for $|x| > a$) ***AG***

[4 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
a.iii. [N/A]
b. [N/A]
c. [N/A]

The functions f and g are defined by $f(x) = 2x + \frac{\pi}{5},\ x \in \mathbb{R}$ and $g(x) = 3\sin x + 4,\ x \in \mathbb{R}$.

a. Show that $g \circ f(x) = 3\sin\left(2x + \frac{\pi}{5}\right) + 4$. [1]

b. Find the range of $g \circ f$. [2]

c. Given that $g \circ f\left(\frac{3\pi}{20}\right) = 7$, find the next value of x, greater than $\frac{3\pi}{20}$, for which $g \circ f(x) = 7$. [2]

d. The graph of $y = g \circ f(x)$ can be obtained by applying four transformations to the graph of $y = \sin x$. State what the four transformations represent geometrically and give the order in which they are applied. [4]

Markscheme

a. $g \circ f(x) = g(f(x))$ ***M1***

$= g\left(2x + \frac{\pi}{5}\right)$

$= 3\sin\left(2x + \frac{\pi}{5}\right) + 4$ ***AG***

[1 mark]

b. since $-1 \leq sin\theta \leq +1$, range is $[1,\ 7]$ ***(R1)A1***

[2 marks]

c. $3\sin\left(2x + \frac{\pi}{5}\right) + 4 = 7 \Rightarrow 2x + \frac{\pi}{5} = \frac{\pi}{2} + 2n\pi \Rightarrow x = \frac{3\pi}{20} + n\pi$ ***(M1)***

so next biggest value is $\frac{23\pi}{20}$ ***A1***

Note: Allow use of period.

[2 marks]

d. **Note:** Transformations can be in any order but see notes below.

stretch scale factor 3 parallel to y axis (vertically) ***A1***

vertical translation of 4 up ***A1***

Note: Vertical translation is $\frac{4}{3}$ up if it occurs before stretch parallel to y axis.

stretch scale factor $\frac{1}{2}$ parallel to x axis (horizontally) ***A1***

horizontal translation of $\frac{\pi}{10}$ to the left ***A1***

Note: Horizontal translation is $\frac{\pi}{5}$ to the left if it occurs before stretch parallel to x axis.

Note: Award ***A1*** for magnitude and direction in each case.

Accept any correct terminology provided that the meaning is clear *eg* shift for translation.

[4 marks]

Total [9 marks]

Examiners report

a. Well done.

b. Generally well done, some used more complicated methods rather than considering the range of sine.

c. Fine if they realised the period was π, not if they thought it was 2π.

d. Typically 3 marks were gained. It was the shift in the axis χ of $\frac{\pi}{10}$ that caused the problem.

Consider the functions $f(x) = \tan x,\ 0 \le\ x < \frac{\pi}{2}$ and $g(x) = \frac{x+1}{x-1},\ x \in \mathbb{R},\ x \ne 1.$

a. Find an expression for $g \circ f(x)$, stating its domain. [2]

b. Hence show that $g \circ f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$. [2]

c. Let $y = g \circ f(x)$, find an exact value for $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point on the graph of $y = g \circ f(x)$ where $x = \frac{\pi}{6}$, expressing your answer in the form $a + b\sqrt{3},\ a,\ b \in \mathbb{Z}$. [6]

d. Show that the area bounded by the graph of $y = g \circ f(x)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{6}$ is $\ln\left(1 + \sqrt{3}\right)$. [6]

Markscheme

a. $g \circ f(x) = \frac{\tan x + 1}{\tan x - 1}$ ***A1***

$x \ne \frac{\pi}{4},\ 0 \le x < \frac{\pi}{2}$ ***A1***

[2 marks]

b. $\frac{\tan x + 1}{\tan x - 1} = \frac{\frac{\sin x}{\cos x} + 1}{\frac{\sin x}{\cos x} - 1}$ ***M1A1***

$= \frac{\sin x + \cos x}{\sin x - \cos x}$ ***AG***

[2 marks]

c. **METHOD 1**

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$ ***M1(A1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(2\sin x\cos x - \cos^2 x - \sin^2 x) - (2\sin x\cos x + \cos^2 x + \sin^2 x)}{\cos^2 x + \sin^2 x - 2\sin x\cos x}$

$= \frac{-2}{1 - \sin 2x}$

Substitute $\frac{\pi}{6}$ into any formula for $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1***

$\frac{-2}{1 - \sin\frac{\pi}{3}}$

$= \frac{-2}{1 - \frac{\sqrt{3}}{2}}$ ***A1***

$= \frac{-4}{2 - \sqrt{3}}$

$= \frac{-4}{2 - \sqrt{3}}\left(\frac{2 + \sqrt{3}}{2 + \sqrt{3}}\right)$ ***M1***

$= \frac{-8 - 4\sqrt{3}}{1} = -8 - 4\sqrt{3}$ ***A1***

METHOD 2

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\tan x - 1)\sec^2 x - (\tan x + 1)\sec^2 x}{(\tan x - 1)^2}$ ***M1A1***

$= \frac{-2\sec^2 x}{(\tan x - 1)^2}$ ***A1***

$$= \frac{-2\sec^2\frac{\pi}{6}}{\left(\tan\frac{\pi}{6}-1\right)^2} = \frac{-2\left(\frac{4}{3}\right)}{\left(\frac{1}{\sqrt{3}}-1\right)^2} = \frac{-8}{\left(1-\sqrt{3}\right)^2}$$ ***M1***

Note: Award ***M1*** for substitution $\frac{\pi}{6}$.

$$\frac{-8}{\left(1-\sqrt{3}\right)^2} = \frac{-8}{\left(4-2\sqrt{3}\right)}\frac{\left(4+2\sqrt{3}\right)}{\left(4+2\sqrt{3}\right)} = -8-4\sqrt{3}$$ ***M1A1***

[6 marks]

d. Area $\left|\int_0^{\frac{\pi}{6}} \frac{\sin x+\cos x}{\sin x-\cos x}\mathrm{d}x\right|$ ***M1***

$= \left|[\ln|\sin x - \cos x|]_0^{\frac{\pi}{6}}\right|$ ***A1***

Note: Condone absence of limits and absence of modulus signs at this stage.

$= \left|\ln\left|\sin\frac{\pi}{6} - \cos\frac{\pi}{6}\right| - \ln|\sin 0 - \cos 0|\right|$ ***M1***

$= \left|\ln\left|\frac{1}{2} - \frac{\sqrt{3}}{2}\right| - 0\right|$

$= \left|\ln\left(\frac{\sqrt{3}-1}{2}\right)\right|$ ***A1***

$= -\ln\left(\frac{\sqrt{3}-1}{2}\right) = \ln\left(\frac{2}{\sqrt{3}-1}\right)$ ***A1***

$= \ln\left(\frac{2}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$ ***M1***

$= \ln\left(\sqrt{3}+1\right)$ ***AG***

[6 marks]

Total [16 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

Consider the function $f(x) = \frac{1}{x^2+3x+2},\ x \in \mathbb{R},\ x \neq -2,\ x \neq -1.$

a.i. Express x^2+3x+2 in the form $(x+h)^2+k$. [1]

a.ii. Factorize x^2+3x+2. [1]

b. Sketch the graph of $f(x)$, indicating on it the equations of the asymptotes, the coordinates of the y-intercept and the local maximum. [5]

c. Show that $\frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{x^2+3x+2}$. [1]

d. Hence find the value of p if $\int_0^1 f(x)\mathrm{d}x = \ln(p)$. [4]

e. Sketch the graph of $y = f(|x|)$. [2]

f. Determine the area of the region enclosed between the graph of $y = f(|x|)$, the x-axis and the lines with equations $x = -1$ and $x = 1$. [3]

Markscheme

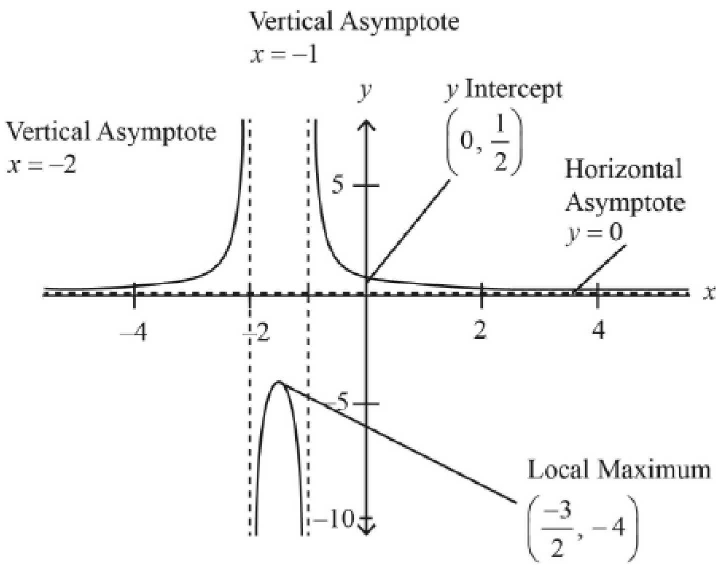
a.i. $x^2 + 3x + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$ ***A1***

[1 mark]

a.ii. $x^2 + 3x + 2 = (x + 2)(x + 1)$ ***A1***

[1 mark]

b.



A1 for the shape

A1 for the equation $y = 0$

A1 for asymptotes $x = -2$ and $x = -1$

A1 for coordinates $\left(-\frac{3}{2},\ -4\right)$

A1 y-intercept $\left(0,\ \frac{1}{2}\right)$

[5 marks]

c. $\frac{1}{x+1} - \frac{1}{x+2} = \frac{(x+2)-(x+1)}{(x+1)(x+2)}$ ***M1***

$= \frac{1}{x^2+3x+2}$ ***AG***

[1 mark]

d. $\int\limits_0^1 \frac{1}{x+1} - \frac{1}{x+2} \mathrm{d}x$

$= [\ln(x+1) - \ln(x+2)]_0^1$ ***A1***

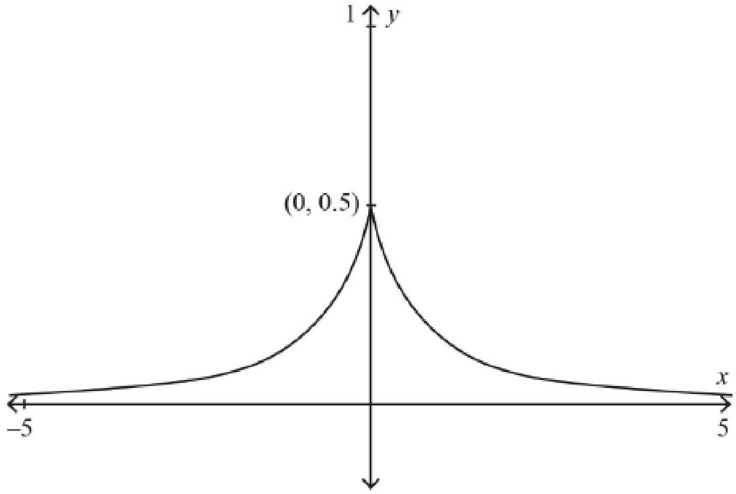
$= \ln 2 - \ln 3 - \ln 1 + \ln 2$ ***M1***

$= \ln\left(\frac{4}{3}\right)$ ***M1A1***

$\therefore p = \frac{4}{3}$

[4 marks]

e.



symmetry about the y-axis ***M1***

correct shape ***A1***

Note: Allow ***FT*** from part (b).

[2 marks]

f. $2\int_0^1 f(x)\mathrm{d}x$ ***(M1)(A1)***

$= 2\ln\left(\frac{4}{3}\right)$ ***A1***

Note: Do not award ***FT*** from part (e).

[3 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]
f. [N/A]

Let $p(x) = 2x^5 + x^4 - 26x^3 - 13x^2 + 72x + 36,\ x \in \mathbb{R}.$

a. For the polynomial equation $p(x) = 0$, state [3]

(i) the sum of the roots;

(ii) the product of the roots.

b. A new polynomial is defined by $q(x) = p(x + 4)$. [2]

Find the sum of the roots of the equation $q(x) = 0$.

Markscheme

a. (i) $\left(-\frac{a_{n-1}}{a_n}=\right)-\frac{1}{2}$ ***A1***

(ii) $\left((-1)^n\frac{a_0}{a_n}=\right)-\frac{36}{2}=(-18)$ ***A1A1***

Note: First ***A1*** is for the negative sign.

[3 marks]

b. **METHOD 1**

if λ satisfies $p(\lambda)=0$ then $q(\lambda-4)=0$

so the roots of $q(x)$ are each 4 less than the roots of $p(x)$ ***(R1)***

so sum of roots is $-\frac{1}{2}-4\times 5=-20.5$ ***A1***

METHOD 2

$p(x+4)=2x^5+2\times 5\times 4x^4\ldots+x^4\ldots=2x^5+41x^4\ldots$ ***(M1)***

so sum of roots is $-\frac{41}{2}=-20.5$ ***A1***

[2 marks]

Tofal [5 marks]

Examiners report

a. Both parts fine if they used the formula, some tried to use the quadratic equivalent formula. Surprisingly some even found all the roots.

b. Some notation problems for weaker candidates. Good candidates used either of the methods shown in the Markscheme.

Let $y(x)=xe^{3x},\ x\in\mathbb{R}$.

a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

b. Prove by induction that $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}=n3^{n-1}\mathrm{e}^{3x}+x3^n\mathrm{e}^{3x}$ for $n\in\mathbb{Z}^+$. [7]

c. Find the coordinates of any local maximum and minimum points on the graph of $y(x)$. [5]

Justify whether any such point is a maximum or a minimum.

d. Find the coordinates of any points of inflexion on the graph of $y(x)$. Justify whether any such point is a point of inflexion. [5]

e. Hence sketch the graph of $y(x)$, indicating clearly the points found in parts (c) and (d) and any intercepts with the axes. [2]

Markscheme

a. $\frac{\mathrm{d}y}{\mathrm{d}x}=1\times\mathrm{e}^{3x}+x\times 3\mathrm{e}^{3x}=\left(\mathrm{e}^{3x}+3x\mathrm{e}^{3x}\right)$ ***M1A1***

[2 marks]

b. let $P(n)$ be the statement $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}=n3^{n-1}\mathrm{e}^{3x}+x3^n\mathrm{e}^{3x}$

prove for $n = 1$ ***M1***

LHS of $P(1)$ is $\frac{\mathrm{d}y}{\mathrm{d}x}$ which is $1 \times \mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x}$ and RHS is $3^0\mathrm{e}^{3x} + x3^1\mathrm{e}^{3x}$ ***R1***

as $\text{LHS} = \text{RHS}$, $P(1)$ is true

assume $P(k)$ is true and attempt to prove $P(k+1)$ is true ***M1***

assuming $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = k3^{k-1}\mathrm{e}^{3x} + x3^k\mathrm{e}^{3x}$

$\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right)$ ***(M1)***

$= k3^{k-1} \times 3\mathrm{e}^{3x} + 1 \times 3^k\mathrm{e}^{3x} + x3^k \times 3\mathrm{e}^{3x}$ ***A1***

$= (k+1)3^k\mathrm{e}^{3x} + x3^{k+1}\mathrm{e}^{3x}$ (as required) ***A1***

Note: Can award the ***A*** marks independent of the ***M*** marks

since $P(1)$ is true and $P(k)$ is true $\Rightarrow P(k+1)$ is true

then (by PMI), $P(n)$ is true $(\forall n \in \mathbb{Z}^+)$ ***R1***

Note: To gain last ***R1*** at least four of the above marks must have been gained.

[7 marks]

c. $\mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x} = 0 \Rightarrow 1 + 3x = 0 \Rightarrow x = -\frac{1}{3}$ ***M1A1***

point is $\left(-\frac{1}{3},\ -\frac{1}{3\mathrm{e}}\right)$ ***A1***

EITHER

$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2 \times 3\mathrm{e}^{3x} + x \times 3^2\mathrm{e}^{3x}$

when $x = -\frac{1}{3}$, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$ therefore the point is a minimum ***M1A1***

OR

x	$-\frac{1}{3}$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$-ve \quad 0 \quad +ve$

nature table shows point is a minimum ***M1A1***

[5 marks]

d. $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2 \times 3\mathrm{e}^{3x} + x \times 3^2\mathrm{e}^{3x}$ ***A1***

$2 \times 3\mathrm{e}^{3x} + x \times 3^2\mathrm{e}^{3x} = 0 \Rightarrow 2 + 3x = 0 \Rightarrow x = -\frac{2}{3}$ ***M1A1***

point is $\left(-\frac{2}{3},\ -\frac{2}{3\mathrm{e}^2}\right)$ ***A1***

x	$-\frac{2}{3}$
$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$	$-ve \quad 0 \quad +ve$

since the curvature does change (concave down to concave up) it is a point of inflection ***R1***

Note: Allow 3^{rd} derivative is not zero at $-\frac{2}{3}$

[5 marks]

e.

(general shape including asymptote and through origin) ***A1***

showing minimum and point of inflection ***A1***

Note: Only indication of position of answers to (c) and (d) required, not coordinates.

[2 marks]

Total [21 marks]

Examiners report

a. Well done.

b. The logic of an induction proof was not known well enough. Many candidates used what they had to prove rather than differentiating what they had assumed. They did not have enough experience in doing Induction proofs.

c. Good, some forgot to test for min/max, some forgot to give the y value.

d. Again quite good, some forgot to check for change in curvature and some forgot the y value.

e. Some accurate sketches, some had all the information from earlier parts but could not apply it. The asymptote was often missed.

The quadratic equation $x^2 - 2kx + (k-1) = 0$ has roots α and β such that $\alpha^2 + \beta^2 = 4$. Without solving the equation, find the possible values of the real number k.

Markscheme

$\alpha + \beta = 2k$ ***A1***

$\alpha\beta = k - 1$ ***A1***

$(\alpha + \beta)^2 = 4k^2 \Rightarrow \alpha^2 + \beta^2 + 2\underbrace{\alpha\beta}_{k-1} = 4k^2$ ***(M1)***

$\alpha^2 + \beta^2 = 4k^2 - 2k + 2$

$\alpha^2 + \beta^2 = 4 \Rightarrow 4k^2 - 2k - 2 = 0$ ***A1***

attempt to solve quadratic ***(M1)***

$k = 1,\ -\frac{1}{2}$ ***A1***

[6 marks]

Examiners report

[N/A]

A given polynomial function is defined as $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$. The roots of the polynomial equation $f(x) = 0$ are consecutive terms of a geometric sequence with a common ratio of $\frac{1}{2}$ and first term 2.

Given that $a_{n-1} = -63$ and $a_n = 16$ find

a. the degree of the polynomial; [4]

b. the value of a_0. [2]

Markscheme

a. the sum of the roots of the polynomial $= \frac{63}{16}$ ***(A1)***

$2\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right) = \frac{63}{16}$ ***M1A1***

Note: The formula for the sum of a geometric sequence must be equated to a value for the ***M1*** to be awarded.

$1 - \left(\frac{1}{2}\right)^n = \frac{63}{64} \Rightarrow \left(\frac{1}{2}\right)^n = \frac{1}{64}$

$n = 6$ ***A1***

[4 marks]

b. $\frac{a_0}{a_n} = 2 \times 1 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{8} \times \frac{1}{16},\ (\mathrm{a}_n = 16)$ ***M1***

$a_0 = 16 \times 2 \times 1 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{8} \times \frac{1}{16}$

$a_0 = 2^{-5}\ \left(= \frac{1}{32}\right)$ ***A1***

[2 marks]

Total [6 marks]

Examiners report

a. [N/A]

b. [N/A]

Consider the function f, where $f(x) = \arcsin(\ln x)$.

(a) Find the domain of f.

(b) Find $f^{-1}(x)$.

Markscheme

(a) $-1 \leqslant \ln x \leqslant 1$ ***(M1)***

$\Rightarrow \frac{1}{\mathrm{e}} \leqslant x \leqslant \mathrm{e}$ ***A1A1***

(b) $y = \arcsin(\ln x) \Rightarrow \ln x = \sin y$ ***(M1)***

$\ln y = \sin x \Rightarrow y = \mathrm{e}^{\sin x}$ ***(M1)***

$\Rightarrow f^{-1}(x) = \mathrm{e}^{\sin x}$ ***A1***

[6 marks]

Examiners report

Very few candidates attempted part (a), and of those that did, few were successful. Part (b) was answered fairly well by most candidates.

---

a. Show that $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1} - \sqrt{n}$ where $n \geq 0,\ n \in \mathbb{Z}$. [2]

b. Hence show that $\sqrt{2} - 1 < \frac{1}{\sqrt{2}}$. [2]

c. Prove, by mathematical induction, that $\sum\limits_{r=1}^{r=n} \frac{1}{\sqrt{r}} > \sqrt{n}$ for $n \geq 2,\ n \in \mathbb{Z}$. [9]

Markscheme

a. $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}+\sqrt{n+1}} \times \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}$ ***M1***

$= \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n}$ ***A1***

$= \sqrt{n+1} - \sqrt{n}$ ***AG***

[2 marks]

b. $\sqrt{2} - 1 = \frac{1}{\sqrt{2}+\sqrt{1}}$ ***A2***

$< \frac{1}{\sqrt{2}}$ ***AG***

[2 marks]

c. consider the case $n = 2$: required to prove that $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$ ***M1***

from part (b) $\frac{1}{\sqrt{2}} > \sqrt{2} - 1$

hence $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$ is true for $n = 2$ ***A1***

now assume true for $n = k : \sum\limits_{r=1}^{r=k} \frac{1}{\sqrt{r}} > \sqrt{k}$ ***M1***

$\frac{1}{\sqrt{1}} + \ldots + \frac{\sqrt{1}}{\sqrt{k}} > \sqrt{k}$

attempt to prove true for $n = k+1 : \frac{1}{\sqrt{1}} + \ldots + \frac{\sqrt{1}}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ ***(M1)***

from assumption, we have that $\frac{1}{\sqrt{1}} + \ldots + \frac{\sqrt{1}}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$ ***M1***

so attempt to show that $\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$ ***(M1)***

EITHER

$\frac{1}{\sqrt{k+1}} > \sqrt{k+1} - \sqrt{k}$ ***A1***

$\frac{1}{\sqrt{k+1}} > \frac{1}{\sqrt{k}+\sqrt{k+1}}$, (from part a), which is true ***A1***

OR

$\sqrt{k}+\frac{1}{\sqrt{k+1}}=\frac{\sqrt{k+1}\sqrt{k+1}}{\sqrt{k+1}}$ ***A1***

$>\frac{\sqrt{k}\sqrt{k+1}}{\sqrt{k+1}}=\sqrt{k+1}$ ***A1***

THEN

so true for $n=2$ and $n=k$ true $\Rightarrow n=k+1$ true. Hence true for all $n\geq 2$ ***R1***

Note: Award ***R1*** only if all previous ***M*** marks have been awarded.

[9 marks]

Total [13 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Let $g(x)=\log_5|2\log_3 x|$. Find the product of the zeros of g .

Markscheme

$g(x)=0$

$\log_5|2\log_3 x|=0$ ***(M1)***

$|2\log_3 x|=1$ ***A1***

$\log_3 x=\pm\frac{1}{2}$ ***(A1)***

$x=3^{\pm\frac{1}{2}}$ ***A1***

so the product of the zeros of g is $3^{\frac{1}{2}}\times 3^{-\frac{1}{2}}=1$ ***A1 N0***

[5 marks]

Examiners report

There were many candidates showing difficulties in manipulating logarithms and the absolute value to solve the equation.

The functions f and g are defined by $f(x)=ax^2+bx+c,\ x\in\mathbb{R}$ and $g(x)=p\sin x+qx+r,\ x\in\mathbb{R}$ where $a,\ b,\ c,\ p,\ q,\ r$ are real constants.

a. Given that f is an even function, show that $b=0$. [2]

b. Given that g is an odd function, find the value of r. [2]

c. The function h is both odd and even, with domain $\mathbb{R}$. [2]

Find $h(x)$.

Markscheme

a. **EITHER**

$f(-x) = f(x)$ ***M1***

$\Rightarrow ax^2 - bx + c = ax^2 + bx + c \Rightarrow 2bx = 0,\ (\forall x \in \mathbb{R})$ ***A1***

OR

y-axis is eqn of symmetry ***M1***

so $\frac{-b}{2a} = 0$ ***A1***

THEN

$\Rightarrow b = 0$ ***AG***

[2 marks]

b. $g(-x) = -g(x) \Rightarrow p\sin(-x) - qx + r = -p\sin x - qx - r$

$\Rightarrow -p\sin x - qx + r = -p\sin x - qx - r$ ***M1***

Note: ***M1*** is for knowing properties of sin.

$\Rightarrow 2r = 0 \Rightarrow r = 0$ ***A1***

Note: In (a) and (b) allow substitution of a particular value of x

[2 marks]

c. $h(-x) = h(x) = -h(x) \Rightarrow 2h(x) = 0 \Rightarrow h(x) = 0,\ (\forall x)$ ***M1A1***

Note: Accept geometrical explanations.

[2 marks]

Total [6 marks]

Examiners report

a. Sometimes backwards working but many correct approaches.

b. Some candidates did not know what odd and even functions were. Correct solutions from those who applied the definition.

c. Some realised: just apply the definitions. Some did very strange things involving f and g.

Consider the polynomial $q(x) = 3x^3 - 11x^2 + kx + 8$.

a. Given that $q(x)$ has a factor $(x - 4)$, find the value of k. [3]

b. Hence or otherwise, factorize $q(x)$ as a product of linear factors. [3]

Markscheme

a. $q(4) = 0$ ***(M1)***

$192 - 176 + 4k + 8 = 0\ (24 + 4k = 0)$ ***A1***

$k = -6$ ***A1***

[3 marks]

b. $3x^3 - 11x^2 - 6x + 8 = (x - 4)(3x^2 + px - 2)$

equate coefficients of x^2: ***(M1)***

$-12 + p = -11$

$p = 1$

$(x - 4)(3x^2 + x - 2)$ ***(A1)***

$(x - 4)(3x - 2)(x + 1)$ ***A1***

Note: Allow part (b) marks if any of this work is seen in part (a).

Note: Allow equivalent methods (*eg*, synthetic division) for the ***M*** marks in each part.

[3 marks]

Examiners report

a. [N/A]
b. [N/A]

Consider the polynomial $P\left(z\right) = z^5 - 10z^2 + 15z - 6,\ z \in \mathbb{C}$.

The polynomial can be written in the form $P(z) = (z - 1)^3(z^2 + bz + c)$.

Consider the function $q\left(x\right) = x^5 - 10x^2 + 15x - 6,\ x \in \mathbb{R}$.

a. Write down the sum and the product of the roots of $P(z) = 0$. [2]

b. Show that $(z - 1)$ is a factor of $P(z)$. [2]

c. Find the value of b and the value of c. [5]

d. Hence find the complex roots of $P(z) = 0$. [3]

e.i. Show that the graph of $y = q(x)$ is concave up for $x > 1$. [3]

e.ii. Sketch the graph of $y = q(x)$ showing clearly any intercepts with the axes. [3]

Markscheme

a. $\text{sum} = 0$ ***A1***

$\text{product} = 6$ ***A1***

[2 marks]

b. $P(1) = 1 - 10 + 15 - 6 = 0$ ***M1A1***

$\Rightarrow (z-1)$ is a factor of $P(z)$ ***AG***

Note: Accept use of division to show remainder is zero.

[2 marks]

c. **METHOD 1**

$(z-1)^3\left(z^2 + bz + c\right) = z^5 - 10z^2 + 15z - 6$ ***(M1)***

by inspection $c = 6$ ***A1***

$(z^3 - 3z^2 + 3z - 1)(z^2 + bz + 6) = z^5 - 10z^2 + 15z - 6$ ***(M1)(A1)***

$b = 3$ ***A1***

METHOD 2

α, β are two roots of the quadratic

$b = -(\alpha + \beta),\ c = \alpha\beta$ ***(A1)***

from part (a) $1 + 1 + 1 + \alpha + \beta = 0$ ***(M1)***

$\Rightarrow b = 3$ ***A1***

$1 \times 1 \times 1 \times \alpha\beta = 6$ ***(M1)***

$\Rightarrow c = 6$ ***A1***

Note: Award ***FT*** if $b = -7$ following through from their sum $= 10$.

METHOD 3

$(z^5 - 10z^2 + 15z - 6) \div (z-1) = z^4 + z^3 + z^2 - 9z + 6$ ***(M1)A1***

Note: This may have been seen in part (b).

$z^4 + z^3 + z^2 - 9z + 6 \div (z-1) = z^3 + 2z^2 + 3z - 6$ ***(M1)***

$z^3 + 2z^2 + 3z - 6 \div (z-1) = z^2 + 3z + 6$ ***A1A1***

[5 marks]

d. $z^2 + 3z + 6 = 0$ ***M1***

$z = \frac{-3 \pm \sqrt{9 - 4 \bullet 6}}{2}$ ***M1***

$= \frac{-3 \pm \sqrt{-15}}{2}$

$z = -\frac{3}{2} \pm \frac{\mathrm{i}\sqrt{15}}{2}$ ***A1***

(or $z = 1$)

Notes: Award the second ***M1*** for an attempt to use the quadratic formula or to complete the square.

Do not award ***FT*** from (c).

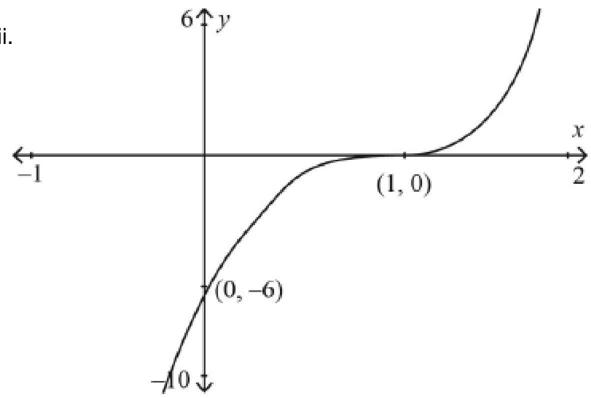
[3 marks]

e.i. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 20x^3 - 20$ ***M1A1***

for $x > 1,\ 20x^3 - 20 > 0 \Rightarrow$ concave up ***R1AG***

[3 marks]

e.ii.



x-intercept at $(1,\ 0)$ ***A1***

y-intercept at $(0,\ -6)$ ***A1***

stationary point of inflexion at $(1,\ 0)$ with correct curvature either side ***A1***

[3 marks]

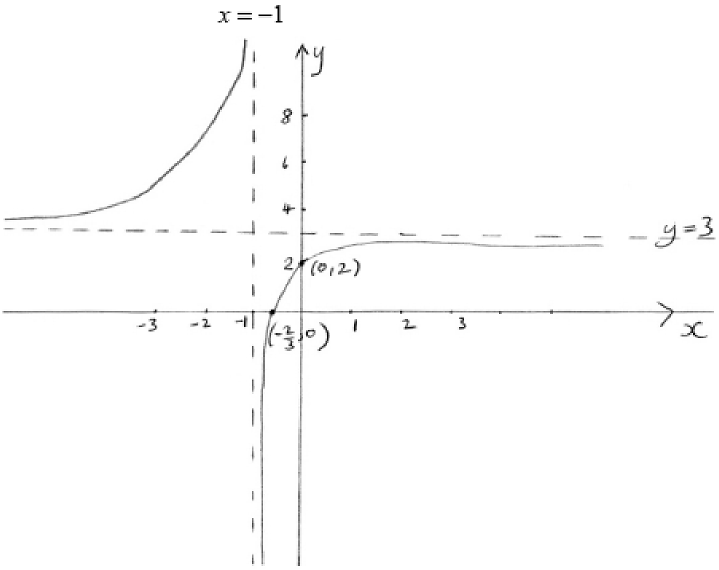
Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e.i. [N/A]
e.ii. [N/A]

The function f is defined as $f(x) = \frac{3x+2}{x+1},\ x \in \mathbb{R},\ x \neq -1.$

Sketch the graph of $y = f(x)$, clearly indicating and stating the equations of any asymptotes and the coordinates of any axes intercepts.

Markscheme



A1A1A1A1A1

Note: Award ***A1*** for correct shape, ***A1*** for $x=-1$ clearly stated and asymptote shown, ***A1*** for $y=3$ clearly stated and asymptote shown, ***A1*** for $\left(-\frac{2}{3},\ 0\right)$ and ***A1*** for $(0,\ 2)$.

[5 marks]

Examiners report

Another standard question. On this occasion, specific coordinates were asked for, so some otherwise good candidates missed out on a couple of marks which they would have gained through greater care.

The cubic equation $x^3+px^2+qx+c=0$, has roots $\alpha,\ \beta,\ \gamma$. By expanding $(x-\alpha)(x-\beta)(x-\gamma)$ show that

a. (i) $p=-(\alpha+\beta+\gamma)$; [3]

(ii) $q=\alpha\beta+\beta\gamma+\gamma\alpha$;

(iii) $c=-\alpha\beta\gamma$.

b. It is now given that $p=-6$ and $q=18$ for parts (b) and (c) below. [5]

(i) In the case that the three roots $\alpha,\ \beta,\ \gamma$ form an arithmetic sequence, show that one of the roots is 2.

(ii) Hence determine the value of c.

c. In another case the three roots $\alpha,\ \beta,\ \gamma$ form a geometric sequence. Determine the value of c. [6]

Markscheme

a. (i)-(iii) given the three roots $\alpha,\ \beta,\ \gamma$, we have

$x^3+px^2+qx+c=(x-\alpha)(x-\beta)(x-\gamma)$ ***M1***

$=\left(x^2-(\alpha+\beta)x+\alpha\beta\right)(x-\gamma)$ ***A1***

$=x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma$ ***A1***

comparing coefficients:

$p = -(\alpha + \beta + \gamma)$ ***AG***

$q = (\alpha\beta + \beta\gamma + \gamma\alpha)$ ***AG***

$c = -\alpha\beta\gamma$ ***AG***

[3 marks]

b. **METHOD 1**

(i) Given $-\alpha - \beta - \gamma = -6$

And $\alpha\beta + \beta\gamma + \gamma\alpha = 18$

Let the three roots be $\alpha,\ \beta,\ \gamma$

So $\beta - \alpha = \gamma - \beta$ ***M1***

or $2\beta = \alpha + \gamma$

Attempt to solve simultaneous equations: ***M1***

$\beta + 2\beta = 6$ ***A1***

$\beta = 2$ ***AG***

(ii) $\alpha + \gamma = 4$

$2\alpha + 2\gamma + \alpha\gamma = 18$

$\Rightarrow \gamma^2 - 4\gamma + 10 = 0$

$\Rightarrow \gamma = \frac{4 \pm \mathrm{i}\sqrt{24}}{2}$ ***(A1)***

Therefore $c = -\alpha\beta\gamma = -\left(\frac{4+\mathrm{i}\sqrt{24}}{2}\right)\left(\frac{4-\mathrm{i}\sqrt{24}}{2}\right)2 = -20$ ***A1***

METHOD 2

(i) let the three roots be $\alpha,\ \alpha - d,\ \alpha + d$ ***M1***

adding roots ***M1***

to give $3\alpha = 6$ ***A1***

$\alpha = 2$ ***AG***

(ii) α is a root, so $2^3 - 6 \times 2^2 + 18 \times 2 + c = 0$ ***M1***

$8 - 24 + 36 + c = 0$

$c = -20$ ***A1***

METHOD 3

(i) let the three roots be $\alpha,\ \alpha - d,\ \alpha + d$ ***M1***

adding roots ***M1***

to give $3\alpha = 6$ ***A1***

$\alpha = 2$ ***AG***

(ii) $q = 18 = 2(2-d) + (2-d)(2+d) + 2(2+d)$ ***M1***

$d^2 = -6 \Rightarrow d = \sqrt{6}\mathrm{i}$

$\Rightarrow c = -20$ ***A1***

[5 marks]

c. **METHOD 1**

Given $-\alpha - \beta - \gamma = -6$

And $\alpha\beta + \beta\gamma + \gamma\alpha = 18$

Let the three roots be $\alpha,\ \beta,\ \gamma$.

So $\frac{\beta}{\alpha} = \frac{\gamma}{\beta}$ ***M1*** or $\beta^2 = \alpha\gamma$

Attempt to solve simultaneous equations: ***M1***

$\alpha\beta + \gamma\beta + \beta^2 = 18$

$\beta(\alpha + \beta + \gamma) = 18$

$6\beta = 18$

$\beta = 3$ ***A1***

$\alpha + \gamma = 3,\ \alpha = \frac{9}{\gamma}$

$\Rightarrow \gamma^2 - 3\gamma + 9 = 0$

$\Rightarrow \gamma = \frac{3 \pm \mathrm{i}\sqrt{27}}{2}$ ***(A1)(A1)***

Therefore $c = -\alpha\beta\gamma = -\left(\frac{3+\mathrm{i}\sqrt{27}}{2}\right)\left(\frac{3-\mathrm{i}\sqrt{27}}{2}\right)3 = -27$ ***A1***

METHOD 2

let the three roots be $a,\ ar,\ ar^2$ ***M1***

attempt at substitution of $a,\ ar,\ ar^2$ and p and q into equations from (a) ***M1***

$6 = a + ar + ar^2\ \left(= a(1 + r + r^2)\right)$ ***A1***

$18 = a^2r + a^2r^3 + a^2r^2\ \left(= a^2r(1 + r + r^2)\right)$ ***A1***

therefore $3 = ar$ ***A1***

therefore $c = -a^3r^3 = -3^3 = -27$ ***A1***

[6 marks]

Total [14 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Consider the following functions:

$h(x) = \arctan(x),\ x \in \mathbb{R}$
$g(x) = \frac{1}{x},\ x \in \mathbb{R},\ x \neq 0$

a. Sketch the graph of $y = h(x)$. [2]

b. Find an expression for the composite function $h \circ g(x)$ and state its domain. [2]

c. Given that $f(x) = h(x) + h \circ g(x)$, [7]

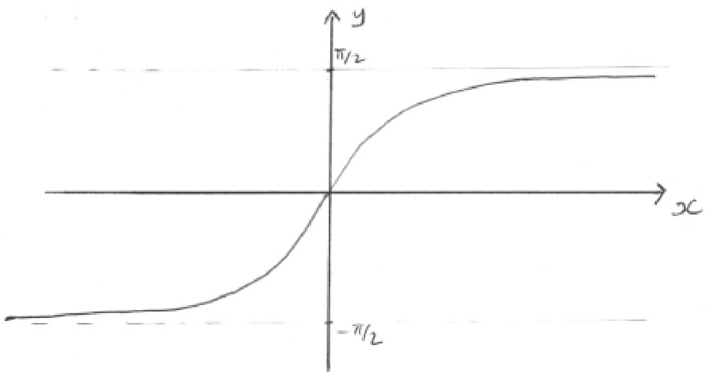
(i) find $f'(x)$ in simplified form;
(ii) show that $f(x) = \frac{\pi}{2}$ for $x > 0$.

d. Nigel states that f is an odd function and Tom argues that f is an even function. [3]

(i) State who is correct and justify your answer.
(ii) Hence find the value of $f(x)$ for $x < 0$.

Markscheme

a.



A1A1

Note: *A1* for correct shape, *A1* for asymptotic behaviour at $y = \pm\frac{\pi}{2}$.

[2 marks]

b. $h \circ g(x) = \arctan\left(\frac{1}{x}\right)$ *A1*

domain of $h \circ g$ is equal to the domain of $g : x \in \circ,\ x \neq 0$ *A1*

[2 marks]

c. (i) $f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right)$

$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times -\frac{1}{x^2}$ *M1A1*

$f'(x) = \frac{1}{1+x^2} + \frac{-\frac{1}{x^2}}{\frac{x^2+1}{x^2}}$ *(A1)*

$= \frac{1}{1+x^2} - \frac{1}{1+x^2}$

$= 0$ *A1*

(ii) **METHOD 1**

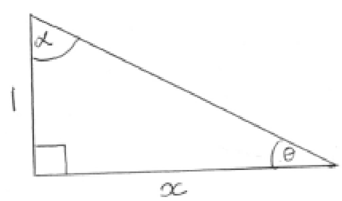
f is a constant *R1*

when $x > 0$

$f(1) = \frac{\pi}{4} + \frac{\pi}{4}$ *M1A1*

$= \frac{\pi}{2}$ *AG*

METHOD 2



from diagram

$\theta = \arctan\frac{1}{x}$ *A1*

$\alpha = \arctan x$ *A1*

$\theta + \alpha = \frac{\pi}{2}$ *R1*

hence $f(x) = \frac{\pi}{2}$ *AG*

METHOD 3

$\tan(f(x)) = \tan\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right)$ *M1*

$= \frac{x+\frac{1}{x}}{1-x\left(\frac{1}{x}\right)}$ *A1*

denominator = 0, so $f(x) = \frac{\pi}{2}$ (for $x > 0$) *R1*

[7 marks]

d. (i) Nigel is correct. *A1*

METHOD 1

$\arctan(x)$ is an odd function and $\frac{1}{x}$ is an odd function

composition of two odd functions is an odd function and sum of two odd functions is an odd function *R1*

METHOD 2

$f(-x) = \arctan(-x) + \arctan\left(-\frac{1}{x}\right) = -\arctan(x) - \arctan\left(\frac{1}{x}\right) = -f(x)$

therefore f is an odd function. ***R1***

(ii) $f(x) = -\frac{\pi}{2}$ ***A1***

[3 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

The function f is defined by $f(x) = 2x^3 + 5,\ -2 \leqslant x \leqslant 2$.

a. Write down the range of f. [2]

b. Find an expression for $f^{-1}(x)$. [2]

c. Write down the domain and range of f^{-1}. [2]

Markscheme

a. $-11 \leqslant f(x) \leqslant 21$ ***A1A1***

Note: ***A1*** for correct end points, ***A1*** for correct inequalities.

[2 marks]

b. $f^{-1}(x) = \sqrt[3]{\frac{x-5}{2}}$ ***(M1)A1***

[2 marks]

c. $-11 \leqslant x \leqslant 21,\ -2 \leqslant f^{-1}(x) \leqslant 2$ ***A1A1***

[2 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Consider the function defined by $f(x) = x\sqrt{1 - x^2}$ on the domain $-1 \leq x \leq 1$.

a. Show that f is an odd function. [2]

b. Find $f'(x)$. [3]

c. Hence find the x-coordinates of any local maximum or minimum points. [3]

d. Find the range of f. [3]

e. Sketch the graph of $y = f(x)$ indicating clearly the coordinates of the x-intercepts and any local maximum or minimum points. [3]

f. Find the area of the region enclosed by the graph of $y = f(x)$ and the x-axis for $x \geq 0$. [4]

g. Show that $\int_{-1}^{1} \left|x\sqrt{1-x^2}\right| \mathrm{d}x > \left|\int_{-1}^{1} x\sqrt{1-x^2}\mathrm{d}x\right|$. [2]

Markscheme

a. $f(-x) = (-x)\sqrt{1-(-x)^2}$ ***M1***

$= -x\sqrt{1-x^2}$

$= -f(x)$ ***R1***

hence f is odd ***AG***

[2 marks]

b. $f'(x) = x \bullet \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \bullet -2x + (1-x^2)^{\frac{1}{2}}$ ***M1A1A1***

[3 marks]

c. $f'(x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \quad \left(= \frac{1-2x^2}{\sqrt{1-x^2}}\right)$ ***A1***

Note: This may be seen in part (b).

Note: Do not allow FT from part (b).

$f'(x) = 0 \Rightarrow 1 - 2x^2 = 0$ ***M1***

$x = \pm\frac{1}{\sqrt{2}}$ ***A1***

[3 marks]

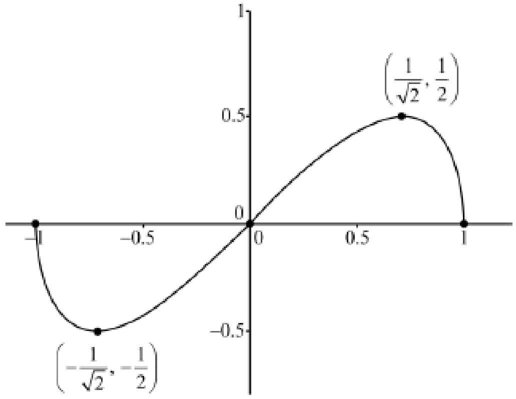
d. y-coordinates of the Max Min Points are $y = \pm\frac{1}{2}$ ***M1A1***

so range of $f(x)$ is $\left[-\frac{1}{2},\ \frac{1}{2}\right]$ ***A1***

Note: Allow FT from (c) if values of x, within the domain, are used.

[3 marks]

e.



Shape: The graph of an odd function, on the given domain, s-shaped,

where the max(min) is the right(left) of 0.5 (-0.5) ***A1***

x-intercepts ***A1***

turning points ***A1***

[3 marks]

f. $\text{area} = \int_0^1 x\sqrt{1-x^2}\mathrm{d}x$ ***(M1)***

attempt at “backwards chain rule” or substitution ***M1***

$= -\frac{1}{2}\int_0^1 (-2x)\sqrt{1-x^2}\mathrm{d}x$

Note: Condone absence of limits for first two marks.

$= \left[\frac{2}{3}(1-x^2)^{\frac{3}{2}} \bullet -\frac{1}{2}\right]_0^1$ ***A1***

$= \left[-\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1$

$= 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$ ***A1***

[4 marks]

g. $\int_{-1}^1 \left|x\sqrt{1-x^2}\right|\mathrm{d}x > 0$ ***R1***

$\left|\int_{-1}^1 x\sqrt{1-x^2}\mathrm{d}x\right| = 0$ ***R1***

so $\int_{-1}^1 \left|x\sqrt{1-x^2}\right|\mathrm{d}x > \left|\int_{-1}^1 x\sqrt{1-x^2}\mathrm{d}x\right|$ ***AG***

[2 marks]

Total [20 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]
f. [N/A]
g. [N/A]

The function f is defined by

$$f(x) = \begin{cases} 2x-1, & x \leqslant 2 \\ ax^2+bx-5, & 2 < x < 3 \end{cases}$$

where a , $b \in \mathbb{R}$.

a. Given that f and its derivative, f' , are continuous for all values in the domain of f, find the values of a and b . [6]

b. Show that f is a one-to-one function. [3]

c. Obtain expressions for the inverse function f^{-1} and state their domains. [5]

Markscheme

a. f continuous $\Rightarrow \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$ ***M1***

$4a+2b=8$ ***A1***

$f'(x) = \begin{cases} 2, & x<2 \\ 2ax+b, & 2<x<3 \end{cases}$ ***A1***

f' continuous $\Rightarrow \lim\limits_{x\to 2^-} f'(x) = \lim\limits_{x\to 2^+} f'(x)$

$4a+b=2$ ***A1***

solve simultaneously ***M1***

to obtain $a = -1$ and $b = 6$ ***A1***

[6 marks]

b. for $x \leqslant 2,\ f'(x) = 2 > 0$ ***A1***

for $2 < x < 3,\ f'(x) = -2x+6 > 0$ ***A1***

since $f'(x) > 0$ for all values in the domain of f, f is increasing ***R1***

therefore one-to-one ***AG***

[3 marks]

c. $x = 2y-1 \Rightarrow y = \frac{x+1}{2}$ ***M1***

$x = -y^2+6y-5 \Rightarrow y^2-6y+x+5=0$ ***M1***

$y = 3 \pm \sqrt{4-x}$

therefore

$f^{-1}(x) = \begin{cases} \frac{x+1}{2}, & x \leqslant 3 \\ 3-\sqrt{4-x}, & 3<x<4 \end{cases}$ ***A1A1A1***

Note: Award ***A1*** for the first line and ***A1A1*** for the second line.

[5 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

The function f is defined by

$$f(x) = \begin{cases} 1 - 2x, & x \le 2 \\ \frac{3}{4}(x-2)^2 - 3, & x > 2 \end{cases}$$

a. Determine whether or not f is continuous. [2]

b. The graph of the function g is obtained by applying the following transformations to the graph of f: [4]

a reflection in the y–axis followed by a translation by the vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Find $g(x)$.

Markscheme

a. $1 - 2(2) = -3$ and $\frac{3}{4}(2-2)^2 - 3 = -3$ ***A1***

both answers are the same, hence f is continuous (at $x = 2$) ***R1***

Note: ***R1*** may be awarded for justification using a graph or referring to limits. Do not award ***A0R1***.

[2 marks]

b. reflection in the y-axis

$$f(-x) = \begin{cases} 1 + 2x, & x \ge -2 \\ \frac{3}{4}(x+2)^2 - 3, & x < -2 \end{cases} \quad \textbf{\textit{(M1)}}$$

Note: Award ***M1*** for evidence of reflecting a graph in y-axis.

translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$g(x) = \begin{cases} 2x - 3, & x \ge 0 \\ \frac{3}{4}x^2 - 3, & x < 0 \end{cases} \quad \textbf{\textit{(M1)A1A1}}$$

Note: Award ***(M1)*** for attempting to substitute $(x-2)$ for x, or translating a graph along positive x-axis.
Award ***A1*** for the correct domains (this mark can be awarded independent of the ***M1***).
Award ***A1*** for the correct expressions.

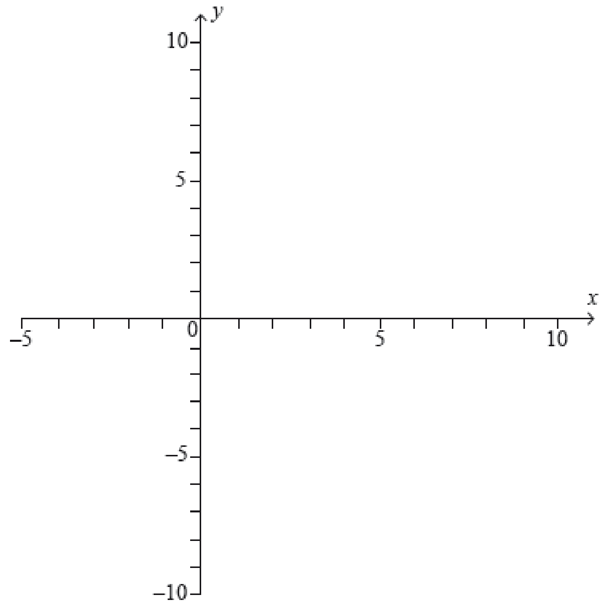
[4 marks]

Examiners report

a. [N/A]
b. [N/A]

---

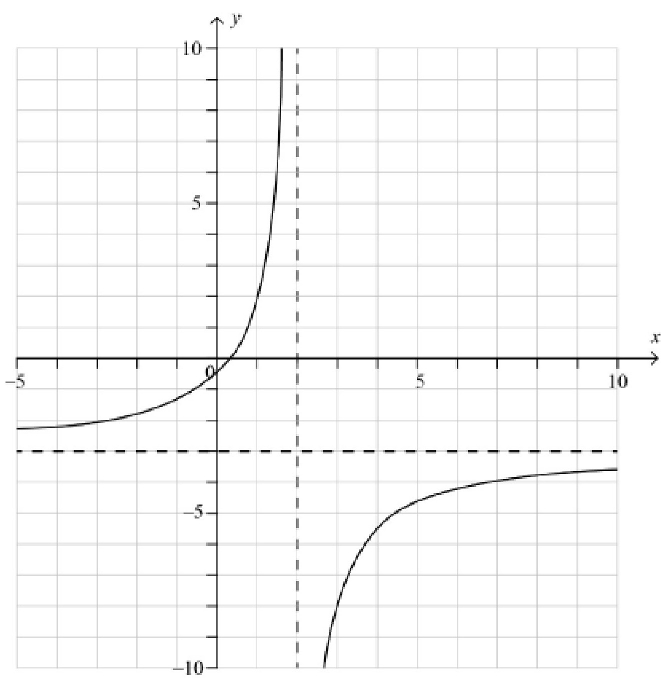
a. Sketch the graph of $y = \frac{1-3x}{x-2}$, showing clearly any asymptotes and stating the coordinates of any points of intersection with the axes. [4]



b. Hence or otherwise, solve the inequality $\left|\frac{1-3x}{x-2}\right| < 2$. [5]

Markscheme

a.



correct vertical asymptote ***A1***

shape including correct horizontal asymptote ***A1***

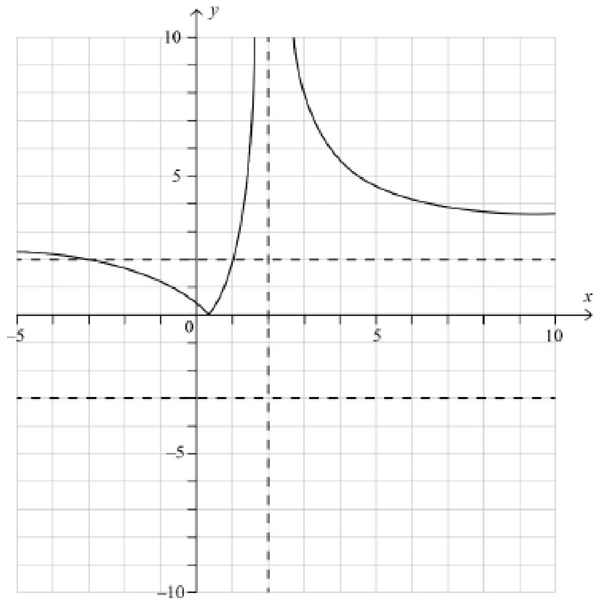
$\left(\frac{1}{3}, 0\right)$ ***A1***

$\left(0, -\frac{1}{2}\right)$ ***A1***

Note: Accept $x = \frac{1}{3}$ and $y = -\frac{1}{2}$ marked on the axes.

[4 marks]

b. **METHOD 1**



$\frac{1-3x}{x-2} = 2$ ***(M1)***

$\Rightarrow x = 1$ ***A1***

$-\left(\frac{1-3x}{x-2}\right) = 2$ ***(M1)***

Note: Award this ***M1*** for the line above or a correct sketch identifying a second critical value.

$\Rightarrow x = -3$ ***A1***

solution is $-3 < x < 1$ ***A1***

METHOD 2

$|1-3x| < 2\,|x-2|\,,\ x \neq 2$

$1 - 6x + 9x^2 < 4(x^2 - 4x + 4)$ ***(M1)A1***

$1 - 6x + 9x^2 < 4x^2 - 16x + 16$

$5x^2 + 10x - 15 < 0$

$x^2 + 2x - 3 < 0$ ***A1***

$(x+3)(x-1) < 0$ ***(M1)***

solution is $-3 < x < 1$ ***A1***

METHOD 3

$-2 < \frac{1-3x}{x-2} < 2$

consider $\frac{1-3x}{x-2} < 2$ ***(M1)***

Note: Also allow consideration of “>” or “=” for the awarding of the ***M*** mark.

recognition of critical value at $x = 1$ ***A1***

consider $-2 < \frac{1-3x}{x-2}$ ***(M1)***

Note: Also allow consideration of “>” or “=” for the awarding of the ***M*** mark.

recognition of critical value at $x = -3$ ***A1***

solution is $-3 < x < 1$ ***A1***

[5 marks]

Examiners report

a. [N/A]
b. [N/A]

The function f is given by $f(x) = x\mathrm{e}^{-x}$ $(x \geqslant 0)$.

a(i)(ii) (i) Find an expression for $f'(x)$. [3]

(ii) Hence determine the coordinates of the point A, where $f'(x) = 0$.

b. Find an expression for $f''(x)$ and hence show the point A is a maximum. [3]

c. Find the coordinates of B, the point of inflexion. [2]

d. The graph of the function g is obtained from the graph of f by stretching it in the *x*-direction by a scale factor 2. [5]

(i) Write down an expression for $g(x)$.

(ii) State the coordinates of the maximum C of g.

(iii) Determine the *x*-coordinates of D and E, the two points where $f(x) = g(x)$.

e. Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same axes, showing clearly the points A, B, C, D and E. [4]

f. Find an exact value for the area of the region bounded by the curve $y = g(x)$, the *x*-axis and the line $x = 1$. [3]

Markscheme

a(i)(ii) (i) $f'(x) = \mathrm{e}^{-x} - x\mathrm{e}^{-x}$ ***M1A1***

(ii) $f'(x) = 0 \Rightarrow x = 1$

coordinates $\left(1,\ \mathrm{e}^{-1}\right)$ ***A1***

[3 marks]

b. $f''(x) = -\mathrm{e}^{-x} - \mathrm{e}^{-x} + x\mathrm{e}^{-x}$ $(= -\mathrm{e}^{-x}(2 - x))$ ***A1***

substituting $x = 1$ into $f''(x)$ ***M1***

$f''(1)$ $\left(= -\mathrm{e}^{-1}\right) < 0$ hence maximum ***R1AG***

[3 marks]

c. $f''(x) = 0$ $(\Rightarrow x = 2)$ ***M1***

coordinates $\left(2,\ 2\mathrm{e}^{-2}\right)$ ***A1***

[2 marks]

d. (i) $g(x) = \frac{x}{2}\mathrm{e}^{-\frac{x}{2}}$ ***A1***

(ii) coordinates of maximum $\left(2,\ \mathrm{e}^{-1}\right)$ ***A1***

(iii) equating $f(x) = g(x)$ and attempting to solve $x\mathrm{e}^{-x} = \frac{x}{2}\mathrm{e}^{-\frac{x}{2}}$

$$\Rightarrow x\left(2\mathrm{e}^{\frac{x}{2}} - \mathrm{e}^{x}\right) = 0 \quad \textit{(A1)}$$

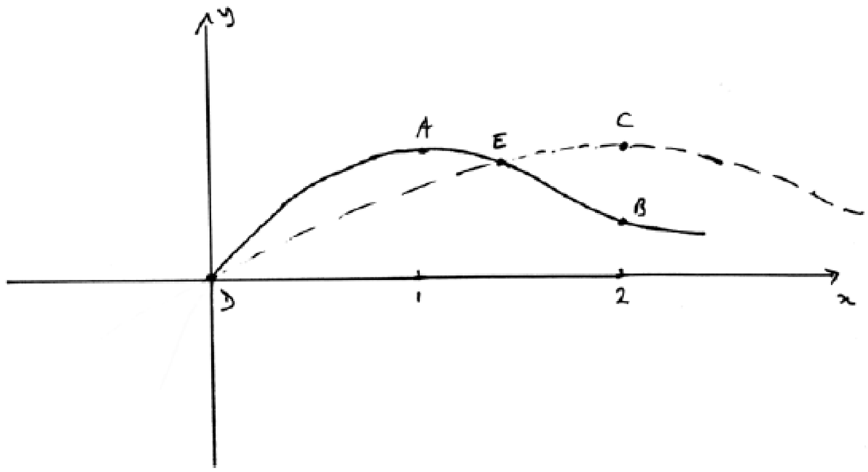
$\Rightarrow x = 0$ ***A1***

or $2e^{\frac{x}{2}} = e^x$

$\Rightarrow e^{\frac{x}{2}} = 2$

$\Rightarrow x = 2\ln 2 \quad (\ln 4)$ ***A1***

Note: Award first (***A1)*** only if factorisation seen or if two correct solutions are seen.

e. 

A4

Note: Award ***A1*** for shape of f, including domain extending beyond $x = 2$.
Ignore any graph shown for $x < 0$.
Award ***A1*** for A and B correctly identified.
Award ***A1*** for shape of g, including domain extending beyond $x = 2$.
Ignore any graph shown for $x < 0$. Allow follow through from f.
Award ***A1*** for C, D and E correctly identified (D and E are interchangeable).

[4 marks]

f. $A = \int_0^1 \frac{x}{2}e^{-\frac{x}{2}}dx$ ***M1***

$= \left[-xe^{-\frac{x}{2}}\right]_0^1 - \int_0^1 -e^{-\frac{x}{2}}dx$ ***A1***

Note: Condone absence of limits or incorrect limits.

$= -e^{-\frac{1}{2}} - \left[2e^{-\frac{x}{2}}\right]_0^1$

$= -e^{-\frac{1}{2}} - \left(2e^{-\frac{1}{2}} - 2\right) = 2 - 3e^{-\frac{1}{2}}$ ***A1***

[3 marks]

Examiners report

a(i)(ii). Part a) proved to be an easy start for the vast majority of candidates.

b. Full marks for part b) were again likewise seen, though a small number shied away from considering the sign of their second derivative, despite the question asking them to do so.

Part c) again proved to be an easily earned 2 marks.

c. Full marks for part b) were again likewise seen, though a small number shied away from considering the sign of their second derivative, despite the question asking them to do so.

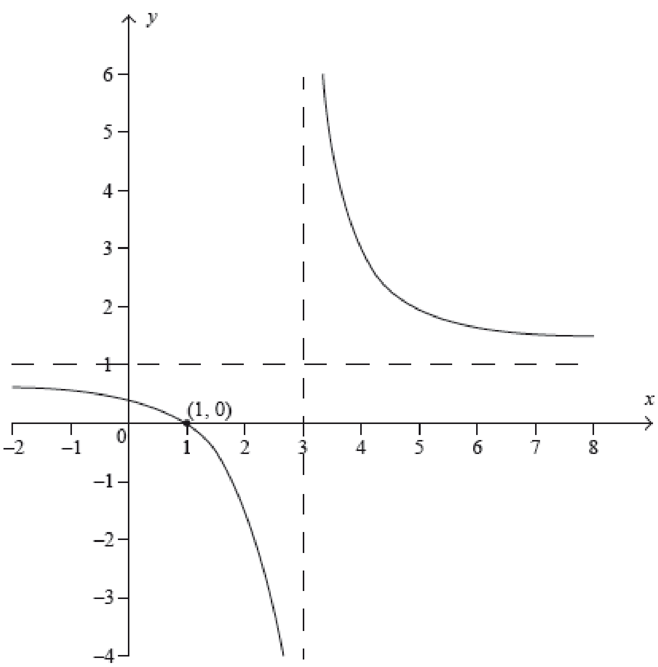
Part c) again proved to be an easily earned 2 marks.

d. Many candidates lost their way in part d). A variety of possibilities for $g(x)$ were suggested, commonly $2x\mathrm{e}^{-2x}$, $\frac{x\mathrm{e}^{-1}}{2}$ or similar variations. Despite section ii) being worth only one mark, (and ‘state’ being present in the question), many laborious attempts at further differentiation were seen. Part diii was usually answered well by those who gave the correct function for $g(x)$.

e. Part e) was also answered well by those who had earned full marks up to that point.

f. While the integration by parts technique was clearly understood, it was somewhat surprising how many careless slips were seen in this part of the question. Only a minority gained full marks for part f).

A rational function is defined by $f(x) = a + \frac{b}{x-c}$ where the parameters $a,\ b,\ c \in \mathbb{Z}$ and $x \in \mathbb{R}\backslash\{c\}$. The following diagram represents the graph of $y = f(x)$.



Using the information on the graph,

a. state the value of a and the value of c; [2]

b. find the value of b. [2]

Markscheme

a. $a = 1$ ***A1***

$c = 3$ ***A1***

[2 marks]

b. use the coordinates of $(1,\ 0)$ on the graph ***M1***

$f(1) = 0 \Rightarrow 1 + \frac{b}{1-3} = 0 \Rightarrow b = 2$ ***A1***

[2 marks]

Examiners report

a. [N/A]
b. [N/A]

The equation $5x^3 + 48x^2 + 100x + 2 = a$ has roots r_1, r_2 and r_3.

Given that $r_1 + r_2 + r_3 + r_1r_2r_3 = 0$, find the value of a.

Markscheme

$r_1 + r_2 + r_3 = \frac{-48}{5}$ ***(M1)(A1)***

$r_1r_2r_3 = \frac{a-2}{5}$ ***(M1)(A1)***

$\frac{-48}{5} + \frac{a-2}{5} = 0$ ***M1***

$a = 50$ ***A1***

Note: Award ***M1A0M1A0M1A1*** if answer of 50 is found using $\frac{48}{5}$ and $\frac{2-a}{5}$.

[6 marks]

Examiners report

[N/A]

Consider the equation $9x^3 - 45x^2 + 74x - 40 = 0$.

a. Write down the numerical value of the sum and of the product of the roots of this equation. [1]

b. The roots of this equation are three consecutive terms of an arithmetic sequence. [6]

Taking the roots to be α , $\alpha \pm \beta$, solve the equation.

Markscheme

a. sum $= \frac{45}{9}$, product $= \frac{40}{9}$ ***A1***

[1 mark]

b. it follows that $3\alpha = \frac{45}{9}$ and $\alpha(\alpha^2 - \beta^2) = \frac{40}{9}$ ***A1A1***

solving, $\alpha = \frac{5}{3}$ ***A1***

$\frac{5}{3}\left(\frac{25}{9} - \beta^2\right) = \frac{40}{9}$ ***M1***

$\beta = (\pm)\frac{1}{3}$ ***A1***

the other two roots are 2, $\frac{4}{3}$ ***A1***

[6 marks]

Examiners report

a. [N/A]
b. [N/A]

Let $f(x) = x^4 + px^3 + qx + 5$ where p, q are constants.

The remainder when $f(x)$ is divided by $(x + 1)$ is 7, and the remainder when $f(x)$ is divided by $(x - 2)$ is 1. Find the value of p and the value of q.

Markscheme

attempt to substitute $x = -1$ or $x = 2$ or to divide polynomials ***(M1)***

$1 - p - q + 5 = 7$, $16 + 8p + 2q + 5 = 1$ or equivalent ***A1A1***

attempt to solve their two equations ***M1***

$p = -3$, $q = 2$ ***A1***

[5 marks]

Examiners report

[N/A]

The cubic polynomial $3x^3 + px^2 + qx - 2$ has a factor $(x + 2)$ and leaves a remainder 4 when divided by $(x + 1)$. Find the value of p and the value of q.

Markscheme

$f(-2) = 0 \ (\Rightarrow -24 + 4p - 2q - 2 = 0)$ ***M1***

$f(-1) = 4 \ (\Rightarrow -3 + p - q - 2 = 4)$ ***M1***

Note: In each case award the ***M*** marks if correct substitution attempted and right-hand side correct.

attempt to solve simultaneously $(2p - q = 13, \ p - q = 9)$ ***M1***

$p = 4$ ***A1***

$q = -5$ ***A1***

[5 marks]

Examiners report

Many candidates scored full marks on what was thought to be an easy first question. However, a number of candidates wrote down two correct equations but proceeded to make algebraic errors and thus found incorrect values for p and q. A small number also attempted to answer this question using long division, but fully correct answers using this technique were rarely seen.

The quadratic equation $2x^2 - 8x + 1 = 0$ has roots α and β.

a. Without solving the equation, find the value of [2]

(i) $\alpha + \beta$;

(ii) $\alpha\beta$.

b. Another quadratic equation $x^2 + px + q = 0,\ p,\ q \in \mathbb{Z}$ has roots $\frac{2}{\alpha}$ and $\frac{2}{\beta}$. [4]

Find the value of p and the value of q.

Markscheme

a. using the formulae for the sum and product of roots:

(i) $\alpha + \beta = 4$ ***A1***

(ii) $\alpha\beta = \frac{1}{2}$ ***A1***

Note: Award ***A0A0*** if the above results are obtained by solving the original equation (except for the purpose of checking).

[2 marks]

b. **METHOD 1**

required quadratic is of the form $x^2 - \left(\frac{2}{\alpha} + \frac{2}{\beta}\right)x + \left(\frac{2}{\alpha}\right)\left(\frac{2}{\beta}\right)$ ***(M1)***

$q = \frac{4}{\alpha\beta}$

$q = 8$ ***A1***

$p = -\left(\frac{2}{\alpha} + \frac{2}{\beta}\right)$

$= -\frac{2(\alpha+\beta)}{\alpha\beta}$ ***M1***

$= -\frac{2\times 4}{\frac{1}{2}}$

$p = -16$ ***A1***

Note: Accept the use of exact roots

METHOD 2

replacing x with $\frac{2}{x}$ ***M1***

$2\left(\frac{2}{x}\right)^2 - 8\left(\frac{2}{x}\right) + 1 = 0$

$\frac{8}{x^2} - \frac{16}{x} + 1 = 0$ ***(A1)***

$x^2 - 16x + 8 = 0$

$p = -16$ and $q = 8$ ***A1A1***

Note: Award ***A1A0*** for $x^2 - 16x + 8 = 0$ *ie*, if $p = -16$ and $q = 8$ are not explicitly stated.

[4 marks]

Total [6 marks]

Examiners report

a. Most candidates obtained full marks.

b. Many candidates obtained full marks, but some responses were inefficiently expressed. A very small minority attempted to use the exact roots, usually unsuccessfully.

Consider the following functions:

$$f(x) = \frac{2x^2 + 3}{75},\ x \geqslant 0$$

$$g(x) = \frac{|3x - 4|}{10},\ x \in \mathbb{R}\ .$$

a. State the range of f and of g . [2]

b. Find an expression for the composite function $f \circ g(x)$ in the form $\frac{ax^2+bx+c}{3750}$, where $a,\ b$ and $c \in \mathbb{Z}$. [4]

c. (i) Find an expression for the inverse function $f^{-1}(x)$. [4]

(ii) State the domain and range of f^{-1}.

d. The domains of f and g are now restricted to $\{0,\ 1,\ 2,\ 3,\ 4\}$. [6]

By considering the values of f and g on this new domain, determine which of f and g could be used to find a probability distribution for a discrete random variable X , stating your reasons clearly.

e. Using this probability distribution, calculate the mean of X . [2]

Markscheme

a. $f(x) \geqslant \frac{1}{25}$ ***A1***

$g(x) \in \mathbb{R},\ g(x) \geqslant 0$ ***A1***

[2 marks]

b. $f \circ g(x) = \frac{2\left(\frac{3x-4}{10}\right)^2+3}{75}$ ***M1A1***

$= \frac{\frac{2(9x^2-24x+16)}{100}+3}{75}$ ***(A1)***

$= \frac{9x^2-24x+166}{3750}$ ***A1***

[4 marks]

c. (i) **METHOD 1**

$y = \frac{2x^2+3}{75}$

$x^2 = \frac{75y-3}{2}$ ***M1***

$x = \sqrt{\frac{75y-3}{2}}$ ***(A1)***

$\Rightarrow f^{-1}(x) = \sqrt{\frac{75x-3}{2}}$ ***A1***

Note: Accept ± in line 3 for the ***(A1)*** but not in line 4 for the ***A1***.
Award the ***A1*** only if written in the form $f^{-1}(x) = $.

METHOD 2

$y = \frac{2x^2+3}{75}$

$x = \frac{2y^2+3}{75}$ ***M1***

$y = \sqrt{\frac{75x-3}{2}}$ ***(A1)***

$\Rightarrow f^{-1}(x) = \sqrt{\frac{75x-3}{2}}$ ***A1***

Note: Accept ± in line 3 for the ***(A1)*** but not in line 4 for the ***A1***.
Award the ***A1*** only if written in the form $f^{-1}(x) =$.

(ii) domain: $x \geqslant \frac{1}{25}$; range: $f^{-1}(x) \geqslant 0$ ***A1***

[4 marks]

d. probabilities from $f(x)$:

X	0	1	2	3	4
$P(X=x)$	$\frac{3}{75}$	$\frac{5}{75}$	$\frac{11}{75}$	$\frac{21}{75}$	$\frac{35}{75}$

A2

Note: Award ***A1*** for one error, ***A0*** otherwise.

probabilities from $g(x)$:

X	0	1	2	3	4
$P(X=x)$	$\frac{4}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{5}{10}$	$\frac{8}{10}$

A2

Note: Award ***A1*** for one error, ***A0*** otherwise.

only in the case of $f(x)$ does $\sum P(X = x) = 1$, hence only $f(x)$ can be used as a probability mass function ***A2***
[6 marks]

e. $E(x) = \sum x \cdot \mathrm{P}(X = x)$ ***M1***

$= \frac{5}{75} + \frac{22}{75} + \frac{63}{75} + \frac{140}{75} = \frac{230}{75} \left(= \frac{46}{15}\right)$ ***A1***
[2 marks]

Examiners report

a. In (a), the ranges were often given incorrectly, particularly the range of g where the modulus signs appeared to cause difficulty. In (b), it was disappointing to see so many candidates making algebraic errors in attempting to determine the expression for $f \circ g(x)$. Many candidates were unable to solve (d) correctly with arithmetic errors and incorrect reasoning often seen. Since the solution to (e) depended upon a correct choice of function in (d), few correct solutions were seen with some candidates even attempting to use integration, inappropriately, to find the mean of X.

b. In (a), the ranges were often given incorrectly, particularly the range of g where the modulus signs appeared to cause difficulty. In (b), it was disappointing to see so many candidates making algebraic errors in attempting to determine the expression for $f \circ g(x)$. Many candidates were unable to solve (d) correctly with arithmetic errors and incorrect reasoning often seen. Since the solution to (e) depended upon a correct choice of function in (d), few correct solutions were seen with some candidates even attempting to use integration, inappropriately, to find the mean of X.

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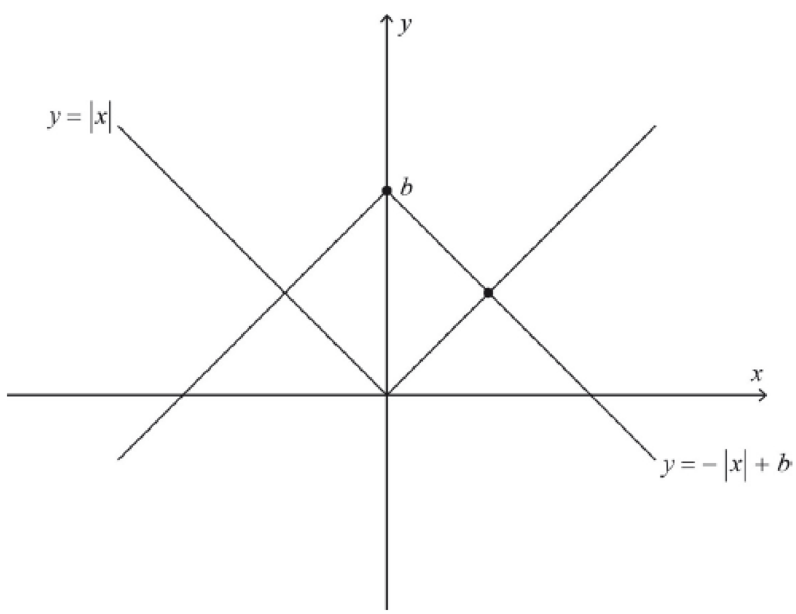
Consider the graphs of $y = |x|$ and $y = -|x| + b$, where $b \in \mathbb{Z}^+$.

a. Sketch the graphs on the same set of axes. [2]

b. Given that the graphs enclose a region of area 18 square units, find the value of *b*. [3]

Markscheme

a.



graphs sketched correctly (condone missing *b*) ***A1A1***

[2 marks]

b. $\frac{b^2}{2} = 18$ ***(M1)A1***

$b = 6$ ***A1***

[3 marks]

Examiners report

a. [N/A]
b. [N/A]

Consider the functions given below.

$$f(x) = 2x + 3$$

$$g(x) = \frac{1}{x}, x \neq 0$$

a. (i) Find $(g \circ f)(x)$ and write down the domain of the function. [2]

(ii) Find $(f \circ g)(x)$ and write down the domain of the function.

b. Find the coordinates of the point where the graph of $y = f(x)$ and the graph of $y = \left(g^{-1} \circ f \circ g\right)(x)$ intersect. [4]

Markscheme

a. (i) $(g \circ f)(x) = \frac{1}{2x+3}, x \neq -\frac{3}{2}$ (or equivalent) ***A1***

(ii) $(f \circ g)(x) = \frac{2}{x} + 3, x \neq 0$ (or equivalent) ***A1***

[2 marks]

b. **EITHER**

$f(x) = \left(g^{-1} \circ f \circ g\right)(x) \Rightarrow (f \circ g)(x)$ ***(M1)***

$\frac{1}{2x+3} = \frac{2}{x} + 3$ ***A1***

OR

$\left(g^{-1} \circ f \circ g\right)(x) = \frac{1}{\frac{2}{x}+3}$ ***A1***

$2x + 3 = \frac{1}{\frac{2}{x}+3}$ ***M1***

THEN

$6x^2 + 12x + 6 = 0$ (or equivalent) ***A1***

$x = -1, y = 1$ (coordinates are $(-1, 1)$) ***A1***

[4 marks]

Examiners report

a. Part (a) was in general well answered and part (b) well attempted. Some candidates had difficulties with the order of composition and in using correct notation to represent the domains of the functions.

b. Part (a) was in general well answered and part (b) well attempted. Some candidates had difficulties with the order of composition and in using correct notation to represent the domains of the functions.

Consider the function $f_n(x) = (\cos 2x)(\cos 4x) \ldots (\cos 2^n x),\ n \in \mathbb{Z}^+$.

a. Determine whether f_n is an odd or even function, justifying your answer. [2]

b. By using mathematical induction, prove that [8]

$$f_n(x) = \frac{\sin 2^{n+1} x}{2^n \sin 2x},\ x \neq \frac{m\pi}{2} \text{ where } m \in \mathbb{Z}.$$

c. Hence or otherwise, find an expression for the derivative of $f_n(x)$ with respect to x. [3]

d. Show that, for $n > 1$, the equation of the tangent to the curve $y = f_n(x)$ at $x = \frac{\pi}{4}$ is $4x - 2y - \pi = 0$. [8]

Markscheme

a. even function ***A1***

since $\cos kx = \cos(-kx)$ **and** $f_n(x)$ is a product of even functions ***R1***

OR

even function ***A1***

since $(\cos 2x)(\cos 4x) \ldots = (\cos(-2x))\,(\cos(-4x)) \ldots$ ***R1***

Note: Do not award ***A0R1***.

[2 marks]

b. consider the case $n = 1$

$\frac{\sin 4x}{2 \sin 2x} = \frac{2 \sin 2x \cos 2x}{2 \sin 2x} = \cos 2x$ ***M1***

hence true for $n = 1$ ***R1***

assume true for $n = k$, *ie*, $(\cos 2x)(\cos 4x) \ldots (\cos 2^k x) = \frac{\sin 2^{k+1} x}{2^k \sin 2x}$ ***M1***

Note: Do not award ***M1*** for "let $n = k$" or "assume $n = k$" or equivalent.

consider $n = k + 1$:

$f_{k+1}(x) = f_k(x)(\cos 2^{k+1} x)$ ***(M1)***

$= \frac{\sin 2^{k+1} x}{2^k \sin 2x} \cos 2^{k+1} x$ ***A1***

$= \frac{2 \sin 2^{k+1} x \cos 2^{k+1} x}{2^{k+1} \sin 2x}$ ***A1***

$= \frac{\sin 2^{k+2} x}{2^{k+1} \sin 2x}$ ***A1***

so $n = 1$ true and $n = k$ true $\Rightarrow n = k + 1$ true. Hence true for all $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, all the previous ***M*** marks must have been awarded.

[8 marks]

c. attempt to use $f' = \frac{vu' - uv'}{v^2}$ (or correct product rule) ***M1***

$f_n'(x) = \frac{(2^n \sin 2x)(2^{n+1} \cos 2^{n+1}x) - (\sin 2^{n+1}x)(2^{n+1} \cos 2x)}{(2^n \sin 2x)^2}$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

[3 marks]

d. $f_n'\left(\frac{\pi}{4}\right) = \frac{\left(2^n \sin \frac{\pi}{2}\right)\left(2^{n+1} \cos 2^{n+1}\frac{\pi}{4}\right) - \left(\sin 2^{n+1}\frac{\pi}{4}\right)\left(2^{n+1} \cos \frac{\pi}{2}\right)}{\left(2^n \sin \frac{\pi}{2}\right)^2}$ ***(M1)(A1)***

$f_n'\left(\frac{\pi}{4}\right) = \frac{(2^n)\left(2^{n+1} \cos 2^{n+1}\frac{\pi}{4}\right)}{(2^n)^2}$ ***(A1)***

$= 2 \cos 2^{n+1}\frac{\pi}{4} \; (= 2 \cos 2^{n-1}\pi)$ ***A1***

$f_n'\left(\frac{\pi}{4}\right) = 2$ ***A1***

$f_n\left(\frac{\pi}{4}\right) = 0$ ***A1***

Note: This ***A*** mark is independent from the previous marks.

$y = 2\left(x - \frac{\pi}{4}\right)$ ***M1A1***

$4x - 2y - \pi = 0$ ***AG***

[8 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

A function *f* is defined by $f(x) = \frac{2x-3}{x-1}, \; x \neq 1$.

(a) Find an expression for $f^{-1}(x)$.

(b) Solve the equation $|f^{-1}(x)| = 1 + f^{-1}(x)$.

Markscheme

(a) **Note:** Interchange of variables may take place at any stage.

for the inverse, solve for *x* in

$y = \frac{2x-3}{x-1}$

$y(x-1) = 2x - 3$ ***M1***

$yx - 2x = y - 3$

$x(y-2) = y - 3$ ***(A1)***

$x = \frac{y-3}{y-2}$

$\Rightarrow f^{-1}(x) = \frac{x-3}{x-2} \quad (x \neq 2)$ ***A1***

Note: Do not award final ***A1*** unless written in the form $f^{-1}(x) = \ldots$

(b) $\pm f^{-1}(x) = 1 + f^{-1}(x)$ leads to

$2\frac{x-3}{x-2} = -1$ ***(M1)***

$x = \frac{8}{3}$ ***A1***

[6 marks]

Examiners report

Many candidates gained the correct answer to part (a), although a significant minority left the answer in the form $y = \ldots$ or $x = \ldots$ rather than $f^{-1}(x) = \ldots$. Only the better candidates were able to make significant progress in part (b).

The quadratic function $f(x) = p + qx - x^2$ has a maximum value of 5 when $x = 3$.

a. Find the value of p and the value of q . [4]

b. The graph of $f(x)$ is translated 3 units in the positive direction parallel to the x-axis. Determine the equation of the new graph. [2]

Markscheme

a. **METHOD 1**

$f'(x) = q - 2x = 0$ ***M1***

$f'(3) = q - 6 = 0$

$q = 6$ ***A1***

$f(3) = p + 18 - 9 = 5$ ***M1***

$p = -4$ ***A1***

METHOD 2

$f(x) = -(x-3)^2 + 5$ ***M1A1***

$= -x^2 + 6x - 4$

$q = 6, p = -4$ ***A1A1***

[4 marks]

b. $g(x) = -4 + 6(x-3) - (x-3)^2 \ (= -31 + 12x - x^2)$ ***M1A1***

Note: Accept any alternative form which is correct.

Award ***M1A0*** for a substitution of $(x + 3)$.

[2 marks]

Examiners report

a. In general candidates handled this question well although a number equated the derivative to the function value rather than zero. Most recognised the shift in the second part although a number shifted only the squared value and not both x values.

b. In general candidates handled this question well although a number equated the derivative to the function value rather than zero. Most recognised the shift in the second part although a number shifted only the squared value and not both x values.

Consider a function f, defined by $f(x) = \frac{x}{2-x}$ for $0 \leqslant x \leqslant 1$.

a. Find an expression for $(f \circ f)(x)$. [3]

b. Let $F_n(x) = \frac{x}{2^n-(2^n-1)x}$, where $0 \leqslant x \leqslant 1$. [8]

Use mathematical induction to show that for any $n \in \mathbb{Z}^+$

$$\underbrace{(f \circ f \circ \ldots \circ f)}_{n\text{ times}}(x) = F_n(x)$$

.

c. Show that $F_{-n}(x)$ is an expression for the inverse of F_n . [6]

d. (i) State $F_n(0)$ and $F_n(1)$. [6]

(ii) Show that $F_n(x) < x$, given $0 < x < 1$, $n \in \mathbb{Z}^+$.

(iii) For $n \in \mathbb{Z}^+$, let A_n be the area of the region enclosed by the graph of F_n^{-1} , the x-axis and the line $x = 1$. Find the area B_n of the region enclosed by F_n and F_n^{-1} in terms of A_n .

Markscheme

a. $(f \circ f)(x) = f\left(\frac{x}{2-x}\right) = \frac{\frac{x}{2-x}}{2-\frac{x}{2-x}}$ ***M1A1***

$(f \circ f)(x) = \frac{x}{4-3x}$ ***A1***

[3 marks]

b. $P(n) : \underbrace{(f \circ f \circ \ldots \circ f)}_{n\text{ times}}(x) = F_n(x)$

$P(1) :\ f(x) = F_1(x)$

$LHS = f(x) = \frac{x}{2-x}$ and $RHS = F_1(x) = \frac{x}{2^1-(2^1-1)x} = \frac{x}{2-x}$ ***A1A1***

$\therefore P(1)$ true

assume that $P(k)$ is true, *i.e.*, $\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k times}}(x) = F_k(x)$ ***M1***

consider $P(k+1)$

EITHER

$$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}}\right)(x) = f\left(F_k(x)\right) \quad \textbf{\textit{(M1)}}$$

$$= f\left(\frac{x}{2^k-(2^k-1)x}\right) = \frac{\frac{x}{2^k-(2^k-1)x}}{2-\frac{x}{2^k-(2^k-1)x}} \quad \textbf{\textit{A1}}$$

$= \frac{x}{2\left(2^k-(2^k-1)x\right)-x} = \frac{x}{2^{k+1}-(2^{k+1}-2)x-x}$ ***A1***

OR

$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}}\right)(x) = F_k(f(x))$ ***(M1)***

$= F_k\left(\frac{x}{2-x}\right) = \frac{\frac{x}{2-x}}{2^k-(2^k-1)\frac{x}{2-x}}$ ***A1***

$= \frac{x}{2^{k+1}-2^kx-2^kx+x}$ ***A1***

THEN

$= \frac{x}{2^{k+1}-(2^{k+1}-1)x} = F_{k+1}(x)$ ***A1***

$P(k)$ true implies $P(k+1)$ true, $P(1)$ true so $P(n)$ true for all $n \in \mathbb{Z}^+$ ***R1***

[8 marks]

c. **METHOD 1**

$x = \frac{y}{2^n-(2^n-1)y} \Rightarrow 2^nx-(2^n-1)xy = y$ ***M1A1***
$\Rightarrow 2^nx = ((2^n-1)x+1)\,y \Rightarrow y = \frac{2^nx}{(2^n-1)x+1}$ ***A1***
$F_n^{-1}(x) = \frac{2^nx}{(2^n-1)x+1}$ ***A1***
$F_n^{-1}(x) = \frac{x}{\frac{2^n-1}{2^n}x+\frac{1}{2^n}}$ ***M1***
$F_n^{-1}(x) = \frac{x}{(1-2^{-n})x+2^{-n}}$ ***A1***
$F_n^{-1}(x) = \frac{x}{2^{-n}-(2^{-n}-1)x}$ ***AG***
METHOD 2
attempt $F_{-n}\left(F_n(x)\right)$ ***M1***
$= F_{-n}\left(\frac{x}{2^n-(2^n-1)x}\right) = \frac{\frac{x}{2^n-(2^n-1)x}}{2^{-n}-(2^{-n}-1)\frac{x}{2^n-(2^n-1)x}}$ ***A1A1***
$= \frac{x}{2^{-n}(2^n-(2^n-1)x)-(2^{-n}-1)x}$ ***A1A1***
Note: Award ***A1*** marks for numerators and denominators.

$= \frac{x}{1} = x$ ***A1AG***
METHOD 3
attempt $F_n\left(F_{-n}(x)\right)$ **M1**
$= F_n\left(\frac{x}{2^{-n}-(2^{-n}-1)x}\right) = \frac{\frac{x}{2^{-n}-(2^{-n}-1)x}}{2^n-(2^n-1)\frac{x}{2^{-n}-(2^{-n}-1)x}}$ ***A1A1***
$= \frac{x}{2^n(2^{-n}-(2^{-n}-1)x)-(2^n-1)x}$ ***A1A1***
Note: Award ***A1*** marks for numerators and denominators.

$= \frac{x}{1} = x$ ***A1AG***
[6 marks]

d. (i) $F_n(0) = 0,\ F_n(1) = 1$ ***A1***

(ii) **METHOD 1**
$2^n-(2^n-1)x-1 = (2^n-1)(1-x)$ ***(M1)***
> 0 if $0 < x < 1$ and $n \in \mathbb{Z}^+$ ***A1***
so $2^n-(2^n-1)x > 1$ and $F_n(x) = \frac{x}{2^n-(2^n-1)x} < \frac{x}{1}(< x)$ ***R1***
$F_n(x) = \frac{x}{2^n-(2^n-1)x} < x$ for $0 < x < 1$ and $n \in \mathbb{Z}^+$ ***AG***
METHOD 2
$\frac{x}{2^n-(2^n-1)x} < x \Leftrightarrow 2^n-(2^n-1)x > 1$ ***(M1)***
$\Leftrightarrow (2^n-1)x < 2^n-1$ ***A1***
$\Leftrightarrow x < \frac{2^n-1}{2^n-1} = 1$ true in the interval $]0,\ 1[$ ***R1***

(iii) $B_n = 2\left(A_n-\frac{1}{2}\right)\ (= 2A_n-1)$ ***(M1)A1***
[6 marks]

Examiners report

a. Part a) proved to be an easy 3 marks for most candidates.

b. Part b) was often answered well, and candidates were well prepared in this session for this type of question. Candidates still need to take care when showing explicitly that P(1) is true, and some are still writing 'Let $n = k$' which gains no marks. The inductive step was often well argued, and given in clear detail, though the final inductive reasoning step was incorrect, or appeared rushed, even from the better candidates. 'True for $n = 1$, $n = k$ and $n = k + 1$' is still disappointingly seen, as were some even more unconvincing variations.

c. Part c) was again very well answered by the majority. A few weaker candidates attempted to find an inverse for the individual case n = 1 , but gained no credit for this.

d. Part d) was not at all well understood, with virtually no candidates able to tie together the hints given by connecting the different parts of the question. Rash, and often thoughtless attempts were made at each part, though by this stage some seemed to be struggling through lack of time. The inequality part of the question tended to be 'fudged', with arguments seen by examiners being largely unconvincing and lacking clarity. A tiny number of candidates provided the correct answer to the final part, though a surprising number persisted with what should have been recognised as fruitless working – usually in the form of long-winded integration attempts.

---

a. State the set of values of a for which the function $x \mapsto \log_a x$ exists, for all $x \in \mathbb{R}^+$. [2]

b. Given that $\log_x y = 4\log_y x$, find all the possible expressions of y as a function of x. [6]

Markscheme

a. $a > 0$ ***A1***

$a \neq 0$ ***A1***

[2 marks]

b. **METHOD 1**

$\log_x y = \frac{\ln y}{\ln x}$ and $\log_y x = \frac{\ln x}{\ln y}$ ***M1A1***

Note: Use of any base is permissible here, not just "e".

$\left(\frac{\ln y}{\ln x}\right)^2 = 4$ ***A1***

$\ln y = \pm 2\ln x$ ***A1***

$y = x^2$ or $\frac{1}{x^2}$ ***A1A1***

METHOD 2

$\log_y x = \frac{\log_x x}{\log_x y} = \frac{1}{\log_x y}$ ***M1A1***

$(\log_x y)^2 = 4$ ***A1***

$\log_x y = \pm 2$ ***A1***

$y = x^2$ or $y = \frac{1}{x^2}$ ***A1A1***

Note: The final two ***A*** marks are independent of the one coming before.

[6 marks]

Total [8 marks]

Examiners report

a. [N/A]
b. [N/A]

When the function $q(x) = x^3 + kx^2 - 7x + 3$ is divided by $(x + 1)$ the remainder is seven times the remainder that is found when the function is divided by $(x + 2)$.

Find the value of k .

Markscheme

$q(-1) = k + 9$ ***M1A1***

$q(-2) = 4k + 9$ ***A1***

$k + 9 = 7(4k + 9)$ ***M1***

$k = -2$ ***A1***

Notes: The first ***M1*** is for one substitution and the consequent equations.
Accept expressions for $q(-1)$ and $q(-2)$ that are not simplified.

[5 marks]

Examiners report

Most candidates were able to access this question although the number who used either synthetic division or long division was surprising as this often lead to difficulty and errors. The most common error was in applying the factor of 7 to the wrong side of the equation. It was also disappointing the number of students who made simple algebraic errors late in the question.

Solve $(\ln x)^2 - (\ln 2)(\ln x) < 2(\ln 2)^2$.

Markscheme

$(\ln x)^2 - (\ln 2)(\ln x) - 2(\ln 2)^2 (= 0)$

EITHER

$\ln x = \frac{\ln 2 \pm \sqrt{(\ln 2)^2 + 8(\ln 2)^2}}{2}$ ***M1***

$= \frac{\ln 2 \pm 3 \ln 2}{2}$ ***A1***

OR

$(\ln x - 2\ln 2)(\ln x + 2\ln 2)\ (= 0)$ ***M1A1***

THEN

$\ln x = 2\ln 2$ or $-\ln 2$ ***A1***

$\Rightarrow x = 4$ or $x = \frac{1}{2}$ ***(M1)A1***

Note: ***(M1)*** is for an appropriate use of a log law in either case, dependent on the previous ***M1*** being awarded, **A1** for both correct answers.

solution is $\frac{1}{2} < x < 4$ ***A1***

[6 marks]

Examiners report

[N/A]

The roots of a quadratic equation $2x^2 + 4x - 1 = 0$ are α and β.

Without solving the equation,

(a) find the value of $\alpha^2 + \beta^2$;

(b) find a quadratic equation with roots α^2 and β^2.

Markscheme

(a) using the formulae for the sum and product of roots:

$\alpha + \beta = -2$ ***A1***

$\alpha\beta = -\frac{1}{2}$ ***A1***

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ ***M1***

$= (-2)^2 - 2\left(-\frac{1}{2}\right)$

$= 5$ ***A1***

Note: Award ***M0*** for attempt to solve quadratic equation.

[4 marks]

(b) $(x - \alpha^2)(x - \beta^2) = x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2$ ***M1***

$x^2 - 5x + \left(-\frac{1}{2}\right)^2 = 0$ ***A1***

$x^2 - 5x + \frac{1}{4} = 0$

Note: Final answer must be an equation. Accept alternative correct forms.

[2 marks]

Total [6 marks]

Examiners report

[N/A]

When the polynomial $3x^3 + ax + b$ is divided by $(x - 2)$, the remainder is 2, and when divided by $(x + 1)$, it is 5. Find the value of a and the value of b.

Markscheme

$P(2) = 24 + 2a + b = 2,\ P(-1) = -3 - a + b = 5$ ***M1A1A1***

$(2a + b = -22,\ \ -a + b = 8)$

Note: Award ***M1*** for substitution of 2 or -1 and equating to remainder, ***A1*** for each correct equation.

attempt to solve simultaneously ***M1***

$a = -10,\ b = -2$ ***A1***

[5 marks]

Examiners report

[N/A]

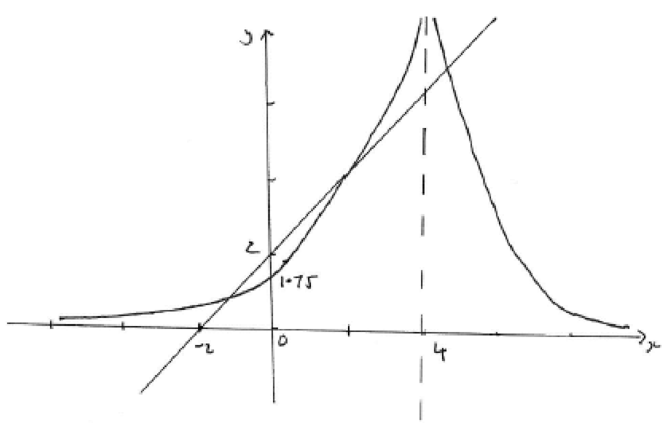
---

a. Sketch on the same axes the curve $y = \left|\frac{7}{x-4}\right|$ and the line $y = x + 2$, clearly indicating any axes intercepts and any asymptotes. [3]

b. Find the exact solutions to the equation $x + 2 = \left|\frac{7}{x-4}\right|$. [5]

Markscheme

a.



A1 for vertical asymptote and for the y-intercept $\frac{7}{4}$

A1 for general shape of $y = \left|\frac{7}{x-4}\right|$ including the x-axis as asymptote

A1 for straight line with y-intercept 2 and x-intercept of -2 ***A1A1A1***

[3 marks]

b. **METHOD 1**

for $x > 4$

$(x+2)(x-4) = 7$ ***(M1)***

$x^2 - 2x - 8 = 7 \Rightarrow x^2 - 2x - 15 = 0$

$(x-5)(x+3) = 0$

$(\text{as } x > 4 \text{ then})\ x = 5$ ***A1***

Note: Award ***A0*** if $x = -3$ is also given as a solution.

for $x < 4$

$(x+2)(x-4) = -7$ ***M1***

$\Rightarrow x^2 - 2x - 1 = 0$

$x = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$ ***(M1)A1***

Note: Second ***M1*** is dependent on first ***M1***.

[5 marks]

METHOD 2

$(x+2)^2 = \frac{49}{(x-4)^2}$ ***M1***

$x^4 - 4x^3 - 12x^2 + 32x + 15 = 0$ ***A1***

$(x+3)(x-5)(x^2 - 2x - 1) = 0$

$x = 5$ ***A1***

Note: Award ***A0*** if $x = -3$ is also given as a solution.

$x = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$ ***(M1)A1***

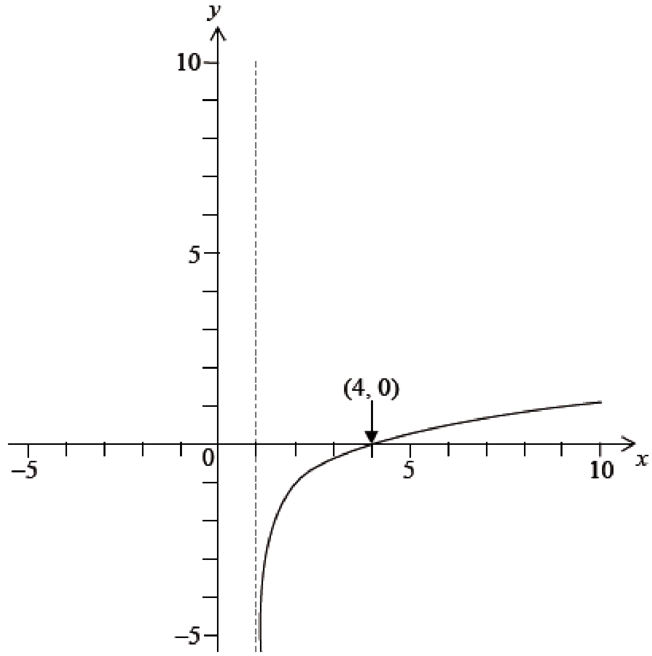
[5 marks]

Examiners report

a. Though generally well done, some candidates lost marks unnecessarily by not heeding the instruction to clearly indicate the axes intercepts and asymptotes.

b. Though this was generally well done, quite a few of the candidates failed to use the graph drawn in part (a) to discount one of the solutions obtained in part (b).

The diagram below shows a sketch of the graph of $y = f(x)$.



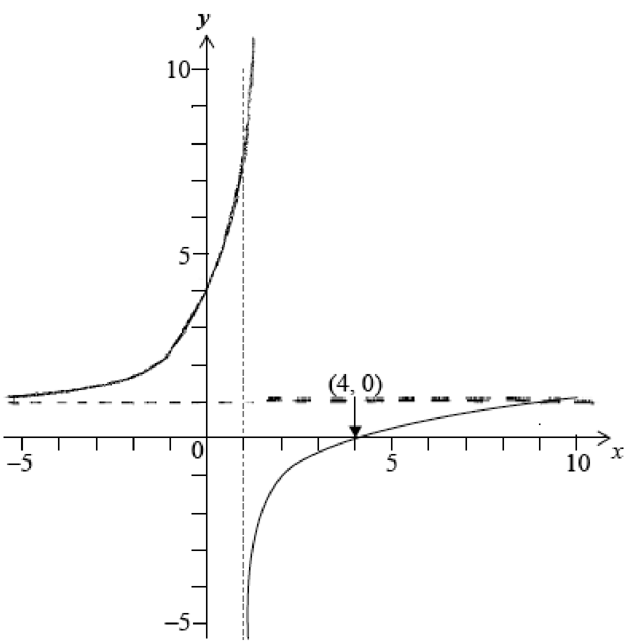
a. Sketch the graph of $y = f^{-1}(x)$ on the same axes. [2]

b. State the range of f^{-1}. [1]

c. Given that $f(x) = \ln(ax + b),\ x > 1$, find the value of a and the value of b. [4]

Markscheme

a. (a)



shape with y-axis intercept (0, 4) ***A1***

Note: Accept curve with an asymptote at $x = 1$ suggested.

correct asymptote $y = 1$ ***A1***

[2 marks]

b. range is $f^{-1}(x) > 1$ (or $]1,\ \infty[$) ***A1***

Note: Also accept $]1,\ 10]$ or $]1,\ 10[$.

Note: Do not allow follow through from incorrect asymptote in (a).

[1 mark]

c. $(4,\ 0) \Rightarrow \ln(4a+b) = 0$ ***M1***

$\Rightarrow 4a+b=1$ ***A1***

asymptote at $x=1 \Rightarrow a+b=0$ ***M1***

$\Rightarrow a=\frac{1}{3},\ b=-\frac{1}{3}$ ***A1***

[4 marks]

Examiners report

a. A number of candidates were able to answer a) and b) correctly but found part c) more challenging. Correct sketches for the inverse were seen, but with a few missing a horizontal asymptote. The range in part b) was usually seen correctly. In part c), only a small number of very good candidates were able to gain full marks. A large number used the point $(4,\ 0)$ to form the equation $4a+b=1$ but were unable (or did not recognise the need) to use the asymptote to form a second equation.

b. A number of candidates were able to answer a) and b) correctly but found part c) more challenging. Correct sketches for the inverse were seen, but with a few missing a horizontal asymptote. The range in part b) was usually seen correctly. In part c), only a small number of very good candidates were able to gain full marks. A large number used the point $(4,\ 0)$ to form the equation $4a+b=1$ but were unable (or did not recognise the need) to use the asymptote to form a second equation.

c. A number of candidates were able to answer a) and b) correctly but found part c) more challenging. Correct sketches for the inverse were seen, but with a few missing a horizontal asymptote. The range in part b) was usually seen correctly. In part c), only a small number of very good candidates were able to gain full marks. A large number used the point $(4,\ 0)$ to form the equation $4a+b=1$ but were unable (or did not recognise the need) to use the asymptote to form a second equation.

Find the set of values of x for which $|x-1| > |2x-1|$.

Markscheme

EITHER

$|x-1| > |2x-1| \Rightarrow (x-1)^2 > (2x-1)^2$ ***M1***

$x^2-2x+1 > 4x^2-4x+1$

$3x^2-2x < 0$ ***A1***

$0 < x < \frac{2}{3}$ ***A1A1*** ***N2***

Note: Award ***A1A0*** for incorrect inequality signs.

OR

$|x-1| > |2x-1|$

$x-1 = 2x-1 \quad x-1 = 1-2x$ ***M1A1***

$-x = 0 \quad 3x = 2$

$x = 0 \quad x = \frac{2}{3}$

Note: Award ***M1*** for any attempt to find a critical value. If graphical methods are used, award ***M1*** for correct graphs, ***A1*** for correct values of *x*.

$0 < x < \frac{2}{3}$ ***A1A1 N2***

Note: Award ***A1A0*** for incorrect inequality signs.

[4 marks]

Examiners report

This question turned out to be more difficult than expected. Candidates who squared both sides or drew a graph generally gave better solutions than those who relied on performing algebraic operations on terms involving modulus signs.

Let $f(x) = \frac{4}{x+2}$, $x \neq -2$ and $g(x) = x - 1$.

If $h = g \circ f$, find

(a) $h(x)$;

(b) $h^{-1}(x)$, where h^{-1} is the inverse of *h*.

Markscheme

(a) $h(x) = g\left(\frac{4}{x+2}\right)$ ***(M1)***

$= \frac{4}{x+2} - 1 \quad \left(= \frac{2-x}{2+x}\right)$ ***A1***

(b) **METHOD 1**

$x = \frac{4}{y+2} - 1$ (interchanging *x* and *y*) ***M1***

Attempting to solve for *y* ***M1***

$(y+2)(x+1) = 4 \quad \left(y + 2 = \frac{4}{x+1}\right)$ ***(A1)***

$h^{-1}(x) = \frac{4}{x+1} - 2 \quad (x \neq -1)$ ***A1 N1***

METHOD 2

$x = \frac{2-y}{2+y}$ (interchanging *x* and *y*) ***M1***

Attempting to solve for *y* ***M1***

$xy + y = 2 - 2x \quad (y(x+1) = 2(1-x))$ ***(A1)***

$h^{-1}(x) = \frac{2(1-x)}{x+1} \quad (x \neq -1)$ ***A1 N1***

Note: In either **METHOD 1** or **METHOD 2** rearranging first and interchanging afterwards is equally acceptable.

[6 marks]

Examiners report

This question was generally well done, with very few candidates calculating $f \circ g$ rather than $g \circ f$.

The function f is given by $f(x) = \frac{3^x+1}{3^x-3^{-x}}$, for $x > 0$.

a. Show that $f(x) > 1$ for all $x > 0$. [3]

b. Solve the equation $f(x) = 4$. [4]

Markscheme

a. **EITHER**

$f(x) - 1 = \frac{1+3^{-x}}{3^x-3^{-x}}$ ***M1A1***

> 0 as both numerator and denominator are positive ***R1***

OR

$3^x + 1 > 3^x > 3^x - 3^{-x}$ ***M1A1***

Note: Accept a convincing valid argument the numerator is greater than the denominator.

numerator and denominator are positive ***R1***

hence $f(x) > 1$ ***AG***

[3 marks]

b. one line equation to solve, for example, $4(3^x - 3^{-x}) = 3^x + 1$, or equivalent ***A1***

$\left(3y^2 - y - 4 = 0\right)$

attempt to solve a three-term equation ***M1***

obtain $y = \frac{4}{3}$ ***A1***

$x = \log_3\left(\frac{4}{3}\right)$ or equivalent ***A1***

Note: Award ***A0*** if an extra solution for x is given.

[4 marks]

Examiners report

a. (a) This is a question where carefully organised reasoning is crucial. It is important to state that both the numerator and the denominator are positive for $x > 0$. Candidates were more successful with part (b) than with part (a).

b. (a) This is a question where carefully organised reasoning is crucial. It is important to state that both the numerator and the denominator are positive for $x > 0$. Candidates were more successful with part (b) than with part (a).

---

a. (i) Express each of the complex numbers $z_1 = \sqrt{3} + \mathrm{i}$, $z_2 = -\sqrt{3} + \mathrm{i}$ and $z_3 = -2\mathrm{i}$ in modulus-argument form. [9]

(ii) Hence show that the points in the complex plane representing z_1, z_2 and z_3 form the vertices of an equilateral triangle.

(iii) Show that $\mathrm{z}_1^{3n} + z_2^{3n} = 2z_3^{3n}$ where $n \in \mathbb{N}$.

b. (i) State the solutions of the equation $z^7 = 1$ for $z \in \mathbb{C}$, giving them in modulus-argument form. [9]

(ii) If w is the solution to $z^7 = 1$ with least positive argument, determine the argument of $1 + w$. Express your answer in terms of π.

(iii) Show that $z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$ is a factor of the polynomial $z^7 - 1$. State the two other quadratic factors with real coefficients.

Markscheme

a. (i) $z_1 = 2\mathrm{cis}\left(\frac{\pi}{6}\right)$, $z_2 = 2\mathrm{cis}\left(\frac{5\pi}{6}\right)$, $z_3 = 2\mathrm{cis}\left(-\frac{\pi}{2}\right)$ or $2\mathrm{cis}\left(\frac{3\pi}{2}\right)$ ***A1A1A1***

Note: Accept modulus and argument given separately, or the use of exponential (Euler) form.

Note: Accept arguments given in rational degrees, except where exponential form is used.

(ii) the points lie on a circle of radius 2 centre the origin ***A1***

differences are all $\frac{2\pi}{3} (\bmod 2\pi)$ ***A1***

$\Rightarrow$ points equally spaced $\Rightarrow$ triangle is equilateral ***R1AG***

Note: Accept an approach based on a clearly marked diagram.

(iii) $\mathrm{z}_1^{3n} + z_2^{3n} = 2^{3n}\mathrm{cis}\left(\frac{n\pi}{2}\right) + 2^{3n}\mathrm{cis}\left(\frac{5n\pi}{2}\right)$ ***M1***

$= 2 \times 2^{3n}\mathrm{cis}\left(\frac{n\pi}{2}\right)$ ***A1***

$2z_3^{3n} = 2 \times 2^{3n}\mathrm{cis}\left(\frac{9n\pi}{2}\right) = 2 \times 2^{3n}\mathrm{cis}\left(\frac{n\pi}{2}\right)$ ***A1AG***

[9 marks]

b. (i) attempt to obtain **seven** solutions in modulus argument form ***M1***

$z = \mathrm{cis}\left(\frac{2k\pi}{7}\right),\ k = 0,\ 1 \ldots 6$ ***A1***

(ii) w has argument $\frac{2\pi}{7}$ and $1 + w$ has argument ϕ,

then $\tan(\phi) = \frac{\sin\left(\frac{2\pi}{7}\right)}{1+\cos\left(\frac{2\pi}{7}\right)}$ ***M1***

$= \frac{2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)}{2\cos^2\left(\frac{\pi}{7}\right)}$ ***A1***

$= \tan\left(\frac{\pi}{7}\right) \Rightarrow \phi = \frac{\pi}{7}$ ***A1***

Note: Accept alternative approaches.

(iii) since roots occur in conjugate pairs, ***(R1)***

$z^7 - 1$ has a quadratic factor $\left(z - \text{cis}\left(\frac{2\pi}{7}\right)\right) \times \left(z - \text{cis}\left(-\frac{2\pi}{7}\right)\right)$ ***A1***

$= z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$ ***AG***

other quadratic factors are $z^2 - 2z\cos\left(\frac{4\pi}{7}\right) + 1$ ***A1***

and $z^2 - 2z\cos\left(\frac{6\pi}{7}\right) + 1$ ***A1***

[9 marks]

Examiners report

a. (i) A disappointingly large number of candidates were unable to give the correct arguments for the three complex numbers. Such errors undermined their efforts to tackle parts (ii) and (iii).

b. Many candidates were successful in part (i), but failed to capitalise on that – in particular, few used the fact that roots of $z^7 - 1 = 0$ come in complex conjugate pairs.

Given the complex numbers $z_1 = 1 + 3\text{i}$ and $z_2 = -1 - \text{i}$.

a. Write down the exact values of $|z_1|$ and $\arg(z_2)$. [2]

b. Find the minimum value of $|z_1 + \alpha z_2|$, where $\alpha \in \mathbb{R}$. [5]

Markscheme

a. $|z_1| = \sqrt{10}$; $\arg(z_2) = -\frac{3\pi}{4}$ $\left(\text{accept } \frac{5\pi}{4}\right)$ ***A1A1***

[2 marks]

b. $|z_1 + \alpha z_2| = \sqrt{(1-\alpha)^2 + (3-\alpha)^2}$ or the squared modulus ***(M1)(A1)***

attempt to minimise $2\alpha^2 - 8\alpha + 10$ or their quadratic or its half or its square root ***M1***

obtain $\alpha = 2$ at minimum ***(A1)***

state $\sqrt{2}$ as final answer ***A1***

[5 marks]

Examiners report

a. Disappointingly, few candidates obtained the correct argument for the second complex number, mechanically using arctan(1) but not thinking about the position of the number in the complex plane.

b. Most candidates obtained the correct quadratic or its square root, but few knew how to set about minimising it.

The same remainder is found when $2x^3 + kx^2 + 6x + 32$ and $x^4 - 6x^2 - k^2x + 9$ are divided by $x + 1$. Find the possible values of k .

Markscheme

let $f(x) = 2x^3 + kx^2 + 6x + 32$
let $g(x) = x^4 - 6x^2 - k^2x + 9$
$f(-1) = -2 + k - 6 + 32 (= 24 + k)$ ***A1***
$g(-1) = 1 - 6 + k^2 + 9 (= 4 + k^2)$ ***A1***
$\Rightarrow 24 + k = 4 + k^2$ ***M1***
$\Rightarrow k^2 - k - 20 = 0$
$\Rightarrow (k - 5)(k + 4) = 0$ ***(M1)***
$\Rightarrow k = 5, -4$ ***A1A1***
[6 marks]

Examiners report

Candidates who used the remainder theorem usually went on to find the two possible values of k. Some candidates, however, attempted to find the remainders using long division. While this is a valid method, the algebra involved proved to be too difficult for most of these candidates.

When $3x^5 - ax + b$ is divided by $x - 1$ and $x + 1$ the remainders are equal. Given that a , $b \in \mathbb{R}$, find

(a) the value of a ;

(b) the set of values of b .

Markscheme

(a) $f(1) = 3 - a + b$ ***(A1)***

$f(-1) = -3 + a + b$ ***(A1)***

$3 - a + b = -3 + a + b$ ***M1***

$2a = 6$

$a = 3$ ***A1*** ***N4***

(b) b is any real number ***A1***

[5 marks]

Examiners report

Many candidates answered part (a) successfully. For part (b), some candidates did not consider that the entire set of real numbers was asked for.

(a) Express the quadratic $3x^2 - 6x + 5$ in the form $a(x+b)^2 + c$, where $a, b, c \in \mathbb{Z}$.

(b) Describe a sequence of transformations that transforms the graph of $y = x^2$ to the graph of $y = 3x^2 - 6x + 5$.

Markscheme

(a) attempt at completing the square ***(M1)***

$3x^2 - 6x + 5 = 3(x^2 - 2x) + 5 = 3(x-1)^2 - 1 + 5$ ***(A1)***

$= 3(x-1)^2 + 2$ ***A1***

$(a = 3,\ b = -1,\ c = 2)$

(b) definition of suitable basic transformations:

$\mathrm{T}_1 =$ stretch in *y* direction scale factor 3 ***A1***

$\mathrm{T}_2 =$ translation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ***A1***

$\mathrm{T}_3 =$ translation $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ ***A1***

[6 marks]

Examiners report

There were fewer correct solutions to this question than might be expected with a significant minority of candidates unable to complete the square successfully and a number of candidates unable to describe the transformations. A minority of candidates knew the correct terminology for the transformations and this potentially highlights the need for teachers to teach students appropriate terminology.

The function f is defined by $f(x) = \frac{1}{x},\ x \neq 0$.

The graph of the function $y = g(x)$ is obtained by applying the following transformations to the graph of $y = f(x)$:

a translation by the vector $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$; a translation by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$;

a. Find an expression for $g(x)$. [2]

b. State the equations of the asymptotes of the graph of g. [2]

Markscheme

a. $g(x) = \frac{1}{x+3} + 1$ ***A1A1***

Note: Award ***A1*** for $x + 3$ in the denominator and ***A1*** for the "$+1$".

[2 marks]

b. $x = -3$ ***A1***

$y = 1$ ***A1***

[2 marks]

Total [4 marks]

Examiners report

a. This question was generally well done. A few candidates made a sign error for the horizontal translation. A few candidates expressed the required equations for the asymptotes as 'inequalities', which received no marks.

b. This question was generally well done. A few candidates made a sign error for the horizontal translation. A few candidates expressed the required equations for the asymptotes as 'inequalities', which received no marks.

---

a. Factorize $z^3 + 1$ into a linear and quadratic factor. [2]

b. Let $\gamma = \frac{1+\mathrm{i}\sqrt{3}}{2}$. [9]

(i) Show that γ is one of the cube roots of -1.

(ii) Show that $\gamma^2 = \gamma - 1$.

(iii) Hence find the value of $(1-\gamma)^6$.

Markscheme

a. using the factor theorem z +1 is a factor ***(M1)***

$z^3 + 1 = (z+1)(z^2 - z + 1)$ ***A1***

[2 marks]

b. (i) **METHOD 1**

$z^3 = -1 \Rightarrow z^3 + 1 = (z+1)(z^2 - z + 1) = 0$ ***(M1)***

solving $z^2 - z + 1 = 0$ ***M1***

$z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \mathrm{i}\sqrt{3}}{2}$ ***A1***

therefore one cube root of -1 is γ ***AG***

METHOD 2

$\gamma^2 = \left(\frac{1+i\sqrt{3}}{2}^2\right) = \frac{-1+i\sqrt{3}}{2}$ ***M1A1***

$\gamma^2 = \frac{-1+i\sqrt{3}}{2} \times \frac{1+i\sqrt{3}}{2} = \frac{-1-3}{4}$ ***A1***

$= -1$ AG

METHOD 3

$\gamma = \frac{1+i\sqrt{3}}{2} = e^{i\frac{\pi}{3}}$ ***M1A1***

$\gamma^3 = e^{i\pi} = -1$ ***A1***

(ii) **METHOD 1**

as γ is a root of $z^2 - z + 1 = 0$ then $\gamma^2 - \gamma + 1 = 0$ ***M1R1***

$\therefore \gamma^2 = \gamma - 1$ ***AG***

Note: Award ***M1*** for the use of $z^2 - z + 1 = 0$ in any way.
Award ***R1*** for a correct reasoned approach.
METHOD 2
$\gamma^2 = \frac{-1+i\sqrt{3}}{2}$ ***M1***
$\gamma - 1 = \frac{1+i\sqrt{3}}{2} - 1 = \frac{-1+i\sqrt{3}}{2}$ ***A1***

(iii) **METHOD 1**
$(1-\gamma)^6 = (-\gamma^2)^6$ ***(M1)***
$= (\gamma)^{12}$ ***A1***
$= (\gamma^3)^4$ ***(M1)***
$= (-1)^4$
$= 1$ ***A1***
METHOD 2
$(1-\gamma)^6$
$= 1 - 6\gamma + 15\gamma^2 - 20\gamma^3 + 15\gamma^4 - 6\gamma^5 + \gamma^6$ ***M1A1***
Note: Award ***M1*** for attempt at binomial expansion.

use of any previous result *e.g.* $= 1 - 6\gamma + 15\gamma^2 + 20 - 15\gamma + 6\gamma^2 + 1$ ***M1***
$= 1$ ***A1***
Note: As the question uses the word 'hence', other methods that do not use previous results are awarded no marks.

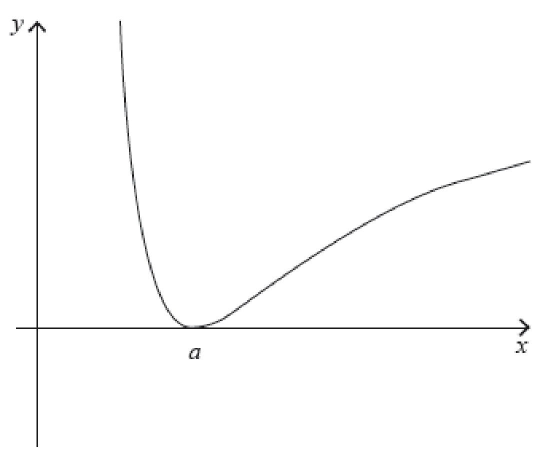
[9 marks]

Examiners report

a. In part a) the factorisation was, on the whole, well done.

b. Part (b) was done well by most although using a substitution method rather than the result above. This used much m retime than was necessary but was successful. A number of candidates did not use the previous results in part (iii) and so seemed to not understand the use of the 'hence'.

The following diagram shows the graph of $y = \frac{(\ln x)^2}{x},\ x > 0$.



The region R is enclosed by the curve, the x-axis and the line $x = e$.

Let $I_n = \int_1^{e} \frac{(\ln x)^n}{x^2} \mathrm{d}x,\ n \in \mathbb{N}$.

a. Given that the curve passes through the point $(a,\ 0)$, state the value of a. [1]

b. Use the substitution $u = \ln x$ to find the area of the region R. [5]

c. (i) Find the value of I_0. [7]

(ii) Prove that $I_n = \frac{1}{\mathrm{e}} + nI_{n-1},\ n \in \mathbb{Z}^+$.

(iii) Hence find the value of I_1.

d. Find the volume of the solid formed when the region R is rotated through 2π about the x-axis. [5]

Markscheme

a. $a = 1$ ***A1***

[1 mark]

b. $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$ ***(A1)***

$\int \frac{(\ln x)^2}{x}\mathrm{d}x = \int u^2 \mathrm{d}u$ ***M1A1***

area $= \left[\frac{1}{3}u^3\right]_0^1$ or $\left[\frac{1}{3}(\ln x)^3\right]_1^{\mathrm{e}}$ ***A1***

$= \frac{1}{3}$ ***A1***

[5 marks]

c. (i) $I_0 = \left[-\frac{1}{x}\right]_1^{\mathrm{e}}$ ***(A1)***

$= 1 - \frac{1}{\mathrm{e}}$ ***A1***

(ii) use of integration by parts ***M1***

$I_n = \left[-\frac{1}{x}(\ln x)^n\right]_1^{\mathrm{e}} + \int_1^{\mathrm{e}} \frac{n(\ln x)^{n-1}}{x^2}\mathrm{d}x$ ***A1A1***

$= -\frac{1}{\mathrm{e}} + nI_{n-1}$ ***AG***

Note: If the substitution $u = \ln x$ is used ***A1A1*** can be awarded for $I_n = [-\mathrm{e}^{-u}u^n]_0^1 + \int_0^1 n\mathrm{e}^{-u}u^{n-1}\mathrm{d}u$.

(iii) $I_1 = -\frac{1}{\mathrm{e}} + 1 \times I_0$ ***(M1)***

$= 1 - \frac{2}{\mathrm{e}}$ ***A1***

[7 marks]

d. (d) volume $= \pi \int_1^{\mathrm{e}} \frac{(\ln x)^4}{x^2}\mathrm{d}x\ (= \pi I_4)$ ***(A1)***

EITHER

$I_4 = -\frac{1}{\mathrm{e}} + 4I_3$ ***M1A1***

$= -\frac{1}{\mathrm{e}} + 4\left(-\frac{1}{\mathrm{e}} + 3I_2\right)$ ***M1***

$= -\frac{5}{\mathrm{e}} + 12I_2 = -\frac{5}{\mathrm{e}} + 12\left(-\frac{1}{\mathrm{e}} + 2I_1\right)$

OR

using parts $\int_1^{\mathrm{e}} \frac{(\ln x)^4}{x^2}\mathrm{d}x = -\frac{1}{\mathrm{e}} + 4\int_1^{\mathrm{e}} \frac{(\ln x)^3}{x^2}\mathrm{d}x$ ***M1A1***

$= -\frac{1}{\mathrm{e}} + 4\left(-\frac{1}{\mathrm{e}} + 3\int_1^{\mathrm{e}} \frac{(\ln x)^2}{x^2}\mathrm{d}x\right)$ ***M1***

THEN

$= -\frac{17}{e} + 24\left(1 - \frac{2}{e}\right) = 24 - \frac{65}{e}$ ***A1***

volume $= \pi\left(24 - \frac{65}{e}\right)$

[5 marks]

Examiners report

a. (a) and (b) were well done. Most candidates could integrate by substitution, though many did not change the limits during the substitution and, though they changed back to x at the end of their solution, under a different markscheme they might have lost marks for this in the intermediate stages.

b. (a) and (b) were well done. Most candidates could integrate by substitution, though many did not change the limits during the substitution and, though they changed back to x at the end of their solution, under a different markscheme they might have lost marks for this in the intermediate stages.

c. (c)(i) This part was well done by the candidates.

   (c)(ii) This proved to be the part that was done by fewest candidates. Those who spotted that they should use integration by parts obtained the answer fairly easily.

   (c)(iii) Many candidates displayed good exam technique in this question and obtained full marks without being able to do part (ii).

d. The same good exam technique was on show here as many students who failed to prove the expression in (c)(ii) were able to use it to obtain full marks in this question. A few candidates failed to remember correctly the formula for a volume of revolution.

Given that $Ax^3 + Bx^2 + x + 6$ is exactly divisible by $(x+1)(x-2)$, find the value of A and the value of B .

Markscheme

using the factor theorem or long division ***(M1)***

$-A + B - 1 + 6 = 0 \Rightarrow A - B = 5$ ***(A1)***

$8A + 4B + 2 + 6 = 0 \Rightarrow 2A + B = -2$ ***(A1)***

$3A = 3 \Rightarrow A = 1$ ***(A1)***

$B = -4$ ***(A1)*** ***(N3)***

Note: Award ***M1A0A0A1A1*** for using $(x-3)$ as the third factor, without justification that the leading coefficient is 1.

[5 marks]

Examiners report

Most candidates attempted this question and it was the best done question on the paper with many fully correct answers. It was good to see a range of approaches used (mainly factor theorem or long division). A number of candidates assumed $(x-3)$ was the missing factor without justification.

The function *f* is defined, for $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, by $f(x) = 2\cos x + x\sin x$.

a. Determine whether *f* is even, odd or neither even nor odd. [3]

b. Show that $f''(0) = 0$. [2]

c. John states that, because $f''(0) = 0$, the graph of *f* has a point of inflexion at the point $(0,\ 2)$. Explain briefly whether John's statement is correct or not. [2]

Markscheme

a. $f(-x) = 2\cos(-x) + (-x)\sin(-x)$ ***M1***

$= 2\cos x + x\sin x \quad (= f(x))$ ***A1***

therefore *f* is even ***A1***

[3 marks]

b. $f'(x) = -2\sin x + \sin x + x\cos x \quad (= -\sin x + x\cos x)$ ***A1***

$f''(x) = -\cos x + \cos x - x\sin x \quad (= -x\sin x)$ ***A1***

so $f''(0) = 0$ ***AG***

[2 marks]

c. John's statement is incorrect because

either; there is a stationary point at (0, 2) and since *f* is an even function and therefore symmetrical about the *y*-axis it must be a maximum or a minimum

or; $f''(x)$ is even and therefore has the same sign either side of (0, 2) ***R2***

[2 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

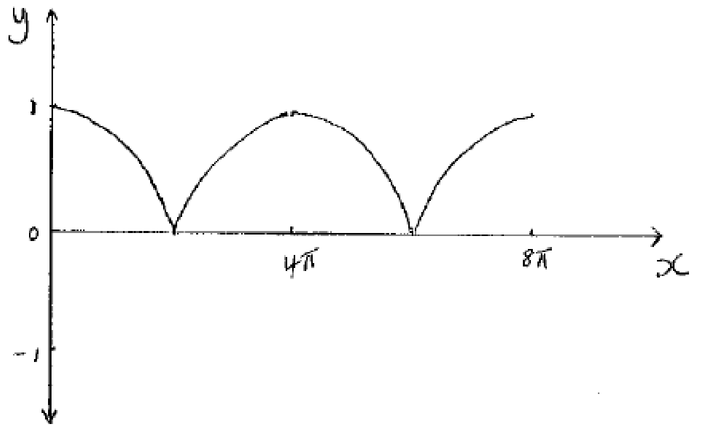
---

a. Sketch the graph of $y = \left|\cos\left(\frac{x}{4}\right)\right|$ for $0 \leqslant x \leqslant 8\pi$. [2]

b. Solve $\left|\cos\left(\frac{x}{4}\right)\right| = \frac{1}{2}$ for $0 \leqslant x \leqslant 8\pi$. [3]

Markscheme

a.



A1A1

Note: Award ***A1*** for correct shape and ***A1*** for correct domain and range.

[2 marks]

b. $\left|\cos\left(\frac{x}{4}\right)\right| = \frac{1}{2}$

$x = \frac{4\pi}{3}$ ***A1***

attempting to find any other solutions ***M1***

Note: Award ***(M1)*** if at least one of the other solutions is correct (in radians or degrees) or clear use of symmetry is seen.

$x = 8\pi - \frac{4\pi}{3} = \frac{20\pi}{3}$

$x = 4\pi - \frac{4\pi}{3} = \frac{8\pi}{3}$

$x = 4\pi + \frac{4\pi}{3} = \frac{16\pi}{3}$ ***A1***

Note: Award ***A1*** for all other three solutions correct and no extra solutions.

Note: If working in degrees, then max ***A0M1A0***.

[3 marks]

Examiners report

a. [N/A]

b. [N/A]

The functions *f* and *g* are defined as:

$$f(x) = e^{x^2},\ x \geqslant 0$$

$$g(x) = \frac{1}{x+3},\ x \neq -3.$$

(a) Find $h(x)$ where $h(x) = g \circ f(x)$.

(b) State the domain of $h^{-1}(x)$.

(c) Find $h^{-1}(x)$.

Markscheme

(a) $h(x) = g \circ f(x) = \frac{1}{e^{x^2}+3}$, $(x \geqslant 0)$ ***(M1)A1***

(b) $0 < x \leqslant \frac{1}{4}$ ***A1A1***

Note: Award ***A1*** for limits and ***A1*** for correct inequality signs.

(c) $y = \frac{1}{e^{x^2}+3}$

$ye^{x^2} + 3y = 1$ ***M1***

$e^{x^2} = \frac{1-3y}{y}$ ***A1***

$x^2 = \ln \frac{1-3y}{y}$ ***M1***

$x = \pm\sqrt{\ln \frac{1-3y}{y}}$

$\Rightarrow h^{-1}(x) = \sqrt{\ln \frac{1-3x}{x}} \left(= \sqrt{\ln\left(\frac{1}{x} - 3\right)}\right)$ ***A1***

[8 marks]

Examiners report

Part (a) was correctly done by the vast majority of candidates. In contrast, only the very best students gave the correct answer to part (b). Part (c) was correctly started by a majority of candidates, but many did not realise that they needed to use logarithms and were careless about the use of notation

---

When $f(x) = x^4 + 3x^3 + px^2 - 2x + q$ is divided by $(x - 2)$ the remainder is 15, and $(x + 3)$ is a factor of $f(x)$.

Find the values of p and q .

Markscheme

$f(2) = 16 + 24 + 4p - 4 + q = 15$ ***M1***

$\Rightarrow 4p + q = -21$ ***A1***

$f(-3) = 81 - 81 + 9p + 6 + q = 0$ ***M1***

$\Rightarrow 9p + q = -6$ ***A1***

$\Rightarrow p = 3$ and $q = -33$ ***A1A1*** ***N0***

[6 marks]

Examiners report

Most candidates made a meaningful attempt at this question. Weaker candidates often made arithmetic errors and a few candidates tried using long division, which also often resulted in arithmetic errors. Overall there were many fully correct solutions.

Solve the following equations:

(a) $\log_2(x-2) = \log_4(x^2-6x+12)$;

(b) $x^{\ln x} = \mathrm{e}^{(\ln x)^3}$.

Markscheme

(a) $\log_2(x-2) = \log_4(x^2-6x+12)$

EITHER

$\log_2(x-2) = \frac{\log_2(x^2-6x+12)}{\log_2 4}$ ***M1***

$2\log_2(x-2) = \log_2(x^2-6x+12)$

OR

$\frac{\log_4(x-2)}{\log_4 2} = \log_4(x^2-6x+12)$ ***M1***

$2\log_4(x-2) = \log_4(x^2-6x+12)$

THEN

$(x-2)^2 = x^2-6x+12$ ***A1***

$x^2-4x+4 = x^2-6x+12$

$x = 4$ ***A1*** ***N1***

[3 marks]

(b) $x^{\ln x} = \mathrm{e}^{(\ln x)^3}$

taking ln of both sides or writing $x = \mathrm{e}^{\ln x}$ ***M1***

$(\ln x)^2 = (\ln x)^3$ ***A1***

$(\ln x)^2(\ln x - 1) = 0$ ***(A1)***

$x = 1,\ x = \mathrm{e}$ ***A1A1*** ***N2***

Note: Award second ***(A1)*** only if factorisation seen or if two correct solutions are seen.

[5 marks]

Total [8 marks]

Examiners report

Part a) was answered well, and a very large proportion of candidates displayed familiarity and confidence with this type of change-of base equation.

In part b), good candidates were able to solve this proficiently. A number obtained only one solution, either through observation or mistakenly cancelling a $\ln x$ term. An incorrect solution $x = \mathrm{e}^3$ was somewhat prevalent amongst the weaker candidates.

The function f is defined by $f(x) = \frac{ax+b}{cx+d}$, for $x \in \mathbb{R},\ x \neq -\frac{d}{c}$.

The function g is defined by $g\left(x\right) = \frac{2x-3}{x-2},\ x \in \mathbb{R},\ x \ne 2$

a. Find the inverse function f^{-1}, stating its domain. [5]

b.i. Express $g\left(x\right)$ in the form $A + \frac{B}{x-2}$ where A, B are constants. [2]

b.ii. Sketch the graph of $y = g\left(x\right)$. State the equations of any asymptotes and the coordinates of any intercepts with the axes. [3]

c. The function h is defined by $h\left(x\right) = \sqrt{x}$, for $x \geq 0$. [4]

State the domain and range of $h \circ g$.

Markscheme

a. attempt to make x the subject of $y = \frac{ax+b}{cx+d}$ ***M1***

$y\left(cx+d\right) = ax + b$ ***A1***

$x = \frac{dy-b}{a-cy}$ ***A1***

$f^{-1}\left(x\right) = \frac{dx-b}{a-cx}$ ***A1***

Note: Do not allow $y =$ in place of $f^{-1}\left(x\right)$.

$x \ne \frac{a}{c},\ \ \left(x \in \mathbb{R}\right)$ ***A1***

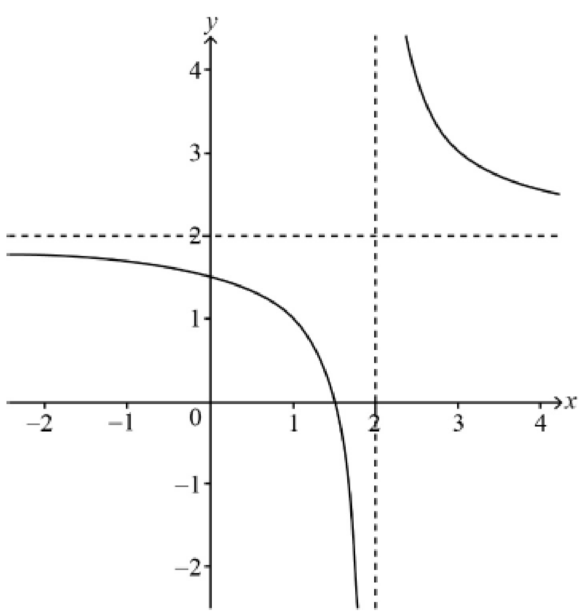
Note: The final ***A*** mark is independent.

[5 marks]

b.i. $g\left(x\right) = 2 + \frac{1}{x-2}$ ***A1A1***

[2 marks]

b.ii.



hyperbola shape, with single curves in second and fourth quadrants and third quadrant blank, including vertical asymptote $x = 2$ ***A1***

horizontal asymptote $y = 2$ ***A1***

intercepts $\left(\frac{3}{2},\ 0\right),\ \left(0,\ \frac{3}{2}\right)$ ***A1***

[3 marks]

c. the domain of $h \circ g$ is $x \leqslant \frac{3}{2},\ x > 2$ ***A1A1*** the range of $h \circ g$ is $y \geqslant 0,\ y \neq \sqrt{2}$ ***A1A1***

[4 marks]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]
c. [N/A]

The function f is defined by $f(x) = \frac{1}{4x^2-4x+5}$.

a. Express $4x^2 - 4x + 5$ in the form $a(x-h)^2 + k$ where $a, h, k \in \mathbb{Q}$. [2]

b. The graph of $y = x^2$ is transformed onto the graph of $y = 4x^2 - 4x + 5$. Describe a sequence of transformations that does this, making the order of transformations clear. [3]

c. Sketch the graph of $y = f(x)$. [2]

d. Find the range of f. [2]

e. By using a suitable substitution show that $\int f(x)\mathrm{d}x = \frac{1}{4}\int \frac{1}{u^2+1}\mathrm{d}u$. [3]

f. Prove that $\int_1^{3.5} \frac{1}{4x^2-4x+5}\mathrm{d}x = \frac{\pi}{16}$. [7]

Markscheme

a. $4(x-0.5)^2 + 4$ ***A1A1***

Note: ***A1*** for two correct parameters, ***A2*** for all three correct.

[2 marks]

b. translation $\begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$ (allow “0.5 to the right”) ***A1***

stretch parallel to y-axis, scale factor 4 (allow vertical stretch or similar) ***A1***

translation $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ (allow “4 up”) ***A1***

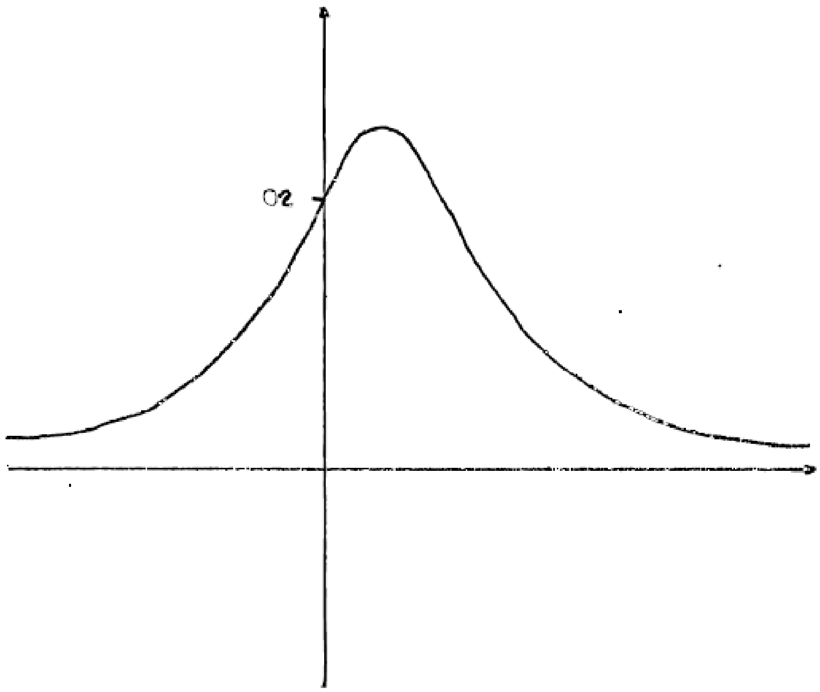
Note: All transformations must state magnitude and direction.

Note: First two transformations can be in either order.

It could be a stretch followed by a single translation of $\begin{pmatrix} 0.5 \\ 4 \end{pmatrix}$. If the vertical translation is before the stretch it is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

[3 marks]

c.



general shape (including asymptote and single maximum in first quadrant), ***A1***

intercept $\left(0, \frac{1}{5}\right)$ or maximum $\left(\frac{1}{2}, \frac{1}{4}\right)$ shown ***A1***

[2 marks]

d. $0 < f(x) \leqslant \frac{1}{4}$ ***A1A1***

Note: ***A1*** for $\leqslant \frac{1}{4}$, ***A1*** for $0 <$.

[2 marks]

e. let $u = x - \frac{1}{2}$ ***A1***

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ (or $\mathrm{d}u = \mathrm{d}x$) ***A1***

$\int \frac{1}{4x^2-4x+5}\mathrm{d}x = \int \frac{1}{4\left(x-\frac{1}{2}\right)^2+4}\mathrm{d}x$ ***A1***

$\int \frac{1}{4u^2+4}\mathrm{d}u = \frac{1}{4}\int \frac{1}{u^2+1}\mathrm{d}u$ ***AG***

Note: If following through an incorrect answer to part (a), do not award final ***A1*** mark.

[3 marks]

f. $\int_1^{3.5} \frac{1}{4x^2-4x+5}\mathrm{d}x = \frac{1}{4}\int_{0.5}^{3} \frac{1}{u^2+1}\mathrm{d}u$ ***A1***

Note: ***A1*** for correct change of limits. Award also if they do not change limits but go back to *x* values when substituting the limit (even if there is an error in the integral).

$\frac{1}{4}[\arctan(u)]_{0.5}^{3}$ ***(M1)***

$\frac{1}{4}\left(\arctan(3) - \arctan\left(\frac{1}{2}\right)\right)$ ***A1***

let the integral $= I$

$\tan 4I = \tan\left(\arctan(3) - \arctan\left(\frac{1}{2}\right)\right)$ ***M1***

$\frac{3-0.5}{1+3\times 0.5} = \frac{2.5}{2.5} = 1$ ***(M1)A1***

$4I = \frac{\pi}{4} \Rightarrow I = \frac{\pi}{16}$ ***A1AG***

[7 marks]

Examiners report

a. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e).

b. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.

(b) Exam technique would have helped those candidates who could not get part (a) correct as any solution of the form given in the question could have led to full marks in part (b). Several candidates obtained expressions which were not of this form in (a) and so were unable to receive any marks in (b) Many missed the fact that if a vertical translation is performed before the vertical stretch it has a different magnitude to if it is done afterwards. Though on this occasion the markscheme was fairly flexible in the words it allowed to be used by candidates to describe the transformations it would be less risky to use the correct expressions.

c. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.

(c) Generally the sketches were poor. The general rule for all sketch questions should be that any asymptotes or intercepts should be clearly labelled. Sketches do not need to be done on graph paper, but a ruler should be used, particularly when asymptotes are involved.

d. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e).

e. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.

(e) and (f) were well done up to the final part of (f), in which candidates did not realise they needed to use the compound angle formula.

f. This question covered many syllabus areas, completing the square, transformations of graphs, range, integration by substitution and compound angle formulae. There were many good solutions to parts (a) – (e) but the following points caused some difficulties.

(e) and (f) were well done up to the final part of (f), in which candidates did not realise they needed to use the compound angle formula.

The function *f* is defined by $f(x) = \frac{2x-1}{x+2}$, with domain $D = \{x : -1 \leqslant x \leqslant 8\}$.

a. Express $f(x)$ in the form $A + \frac{B}{x+2}$, where A and $B \in \mathbb{Z}$. [2]

b. Hence show that $f'(x) > 0$ on D. [2]

c. State the range of *f*. [2]

d. (i) Find an expression for $f^{-1}(x)$. [8]

(ii) Sketch the graph of $y = f(x)$, showing the points of intersection with both axes.

(iii) On the same diagram, sketch the graph of $y = f'(x)$.

e. (i) On a different diagram, sketch the graph of $y = f(|x|)$ where $x \in D$. [7]

(ii) Find all solutions of the equation $f(|x|) = -\frac{1}{4}$.

Markscheme

a. by division or otherwise

$f(x) = 2 - \frac{5}{x+2}$ ***A1A1***

[2 marks]

b. $f'(x) = \frac{5}{(x+2)^2}$ ***A1***

> 0 as $(x+2)^2 > 0$ (on D) ***R1AG***

Note: Do not penalise candidates who use the original form of the function to compute its derivative.

[2 marks]

c. $S = \left[-3, \frac{3}{2}\right]$ ***A2***

Note: Award ***A1A0*** for the correct endpoints and an open interval.

[2 marks]

d. (i) **EITHER**

rearrange $y = f(x)$ to make *x* the subject ***M1***

obtain one-line equation, *e.g.* $2x - 1 = xy + 2y$ ***A1***

$x = \frac{2y+1}{2-y}$ ***A1***

OR

interchange *x* and *y* ***M1***

obtain one-line equation, *e.g.* $2y - 1 = xy + 2x$ ***A1***

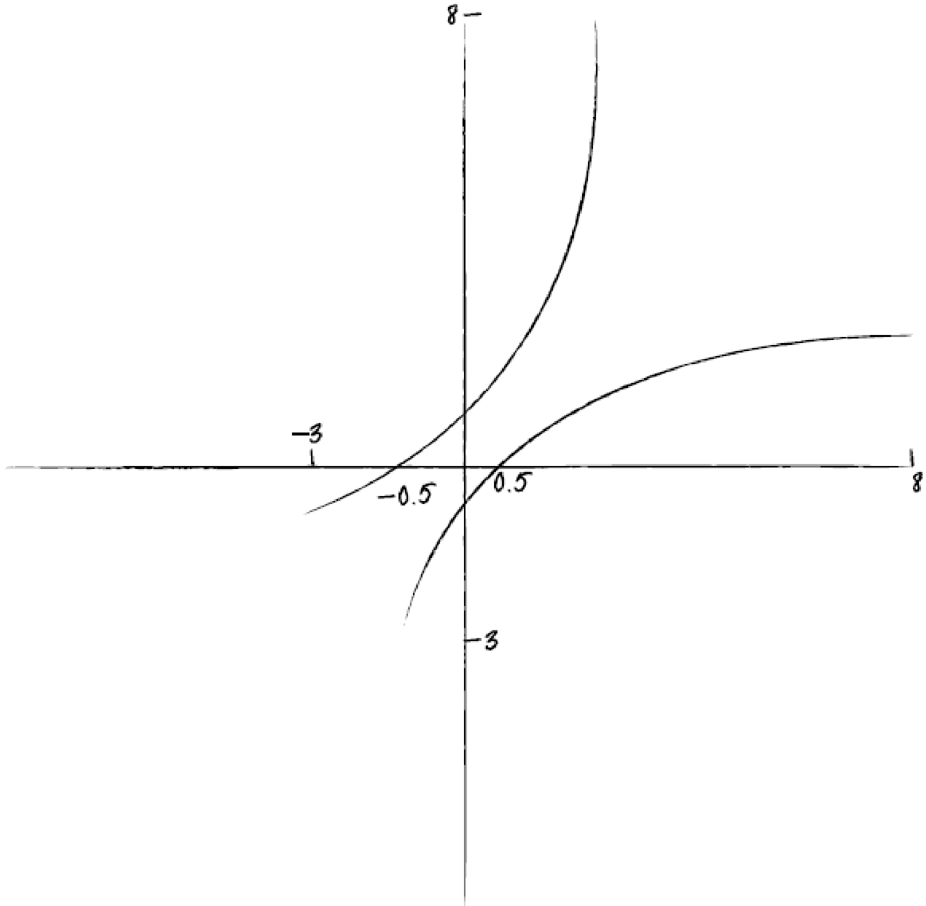
$y = \frac{2x+1}{2-x}$ ***A1***

THEN

$f^{-1}(x) = \frac{2x+1}{2-x}$ ***A1***

Note: Accept $\frac{5}{2-x} - 2$

(ii), (iii)



A1A1A1A1

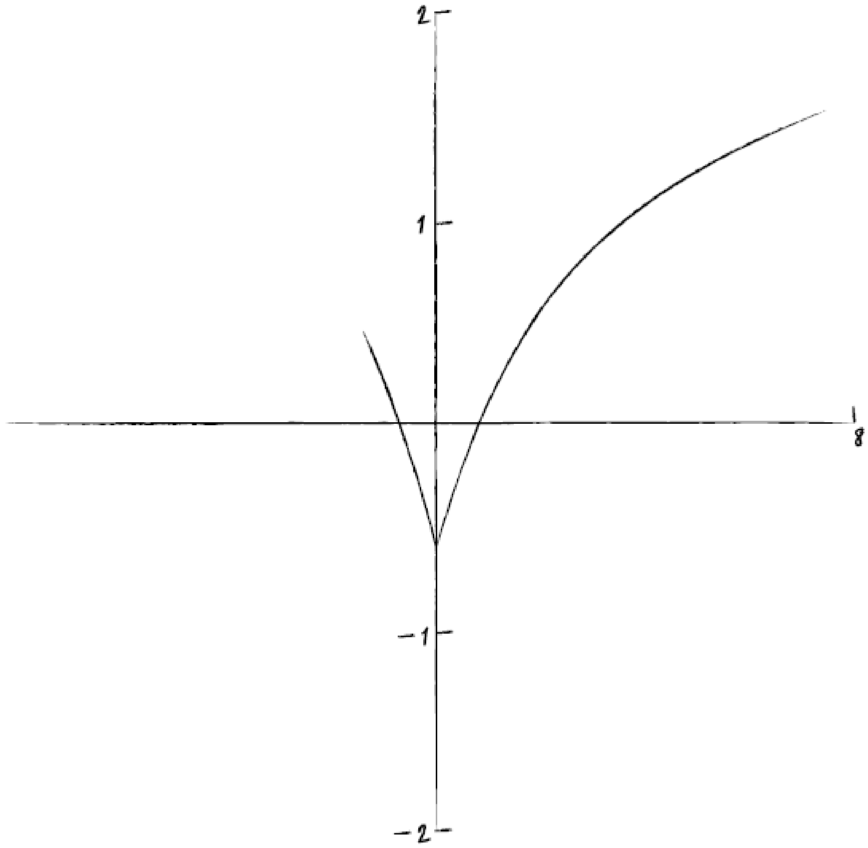
[8 marks]

Note: Award ***A1*** for correct shape of $y = f(x)$.

Award ***A1*** for x intercept $\frac{1}{2}$ seen. Award ***A1*** for y intercept $-\frac{1}{2}$ seen.

Award ***A1*** for the graph of $y = f^{-1}(x)$ being the reflection of $y = f(x)$ in the line $y = x$. Candidates are not required to indicate the full domain, but $y = f(x)$ should not be shown approaching $x = -2$. Candidates, in answering (iii), can **FT** on their sketch in (ii).

e. (i)



A1A1A1

Note: *A1* for correct sketch $x > 0$, *A1* for symmetry, *A1* for correct domain (from –1 to +8).

Note: Candidates can **FT** on their sketch in (d)(ii).

(ii) attempt to solve $f(x) = -\frac{1}{4}$ *(M1)*

obtain $x = \frac{2}{9}$ *A1*

use of symmetry or valid algebraic approach *(M1)*

obtain $x = -\frac{2}{9}$ *A1*

[7 marks]

Examiners report

a. Generally well done.

b. In their answers to Part (b), most candidates found the derivative, but many assumed it was obviously positive.

c. [N/A]

d. Part (d)(i) Generally well done, but some candidates failed to label their final expression as $f^{-1}(x)$. Part (d)(ii) Marks were lost by candidates who failed to mark the intercepts with values.

e. Marks were also lost in this part and in part (e)(i) for graphs that went beyond the explicitly stated domain.

The function f is defined as $f(x) = \mathrm{e}^{3x+1},\ x \in \mathbb{R}$.

a. (i) Find $f^{-1}(x)$. [4]

(ii) State the domain of f^{-1}.

b. The function g is defined as $g(x) = \ln x,\ x \in \mathbb{R}^{+}$. [5]

The graph of $y = g(x)$ and the graph of $y = f^{-1}(x)$ intersect at the point P.

Find the coordinates of P.

c. The graph of $y = g(x)$ intersects the x-axis at the point Q. [3]

Show that the equation of the tangent T to the graph of $y = g(x)$ at the point Q is $y = x - 1$.

d. A region R is bounded by the graphs of $y = g(x)$, the tangent T and the line $x = \mathrm{e}$. [5]

Find the area of the region R.

e. A region R is bounded by the graphs of $y = g(x)$, the tangent T and the line $x = \mathrm{e}$. [6]

(i) Show that $g(x) \le x - 1,\ x \in \mathbb{R}^{+}$.

(ii) By replacing x with $\frac{1}{x}$ in part (e)(i), show that $\frac{x-1}{x} \le g(x),\ x \in \mathbb{R}^{+}$.

Markscheme

a. (i) $x = \mathrm{e}^{3y+1}$ ***M1***

Note: The ***M1*** is for switching variables and can be awarded at any stage.

Further marks do not rely on this mark being awarded.

taking the natural logarithm of both sides and attempting to transpose ***M1***

$\left(f^{-1}(x)\right) = \frac{1}{3}(\ln x - 1)$ ***A1***

(ii) $x \in \mathbb{R}^{+}$ or equivalent, for example $x > 0$. ***A1***

[4 marks]

b. $\ln x = \frac{1}{3}(\ln x - 1) \Rightarrow \ln x - \frac{1}{3}\ln x = -\frac{1}{3}$ (or equivalent) ***M1A1***

$\ln x = -\frac{1}{2}$ (or equivalent) ***A1***

$x = \mathrm{e}^{-\frac{1}{2}}$ ***A1***

coordinates of P are $\left(\mathrm{e}^{-\frac{1}{2}},\ -\frac{1}{2}\right)$ ***A1***

[5 marks]

c. coordinates of Q are $(1, 0)$ seen anywhere ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ ***M1***

at $\mathrm{Q},\ \frac{\mathrm{d}y}{\mathrm{d}x} = 1$ ***A1***

$y = x - 1$ ***AG***

[3 marks]

d. let the required area be A

$A=\int_1^e x-1\mathrm{d}x-\int_1^e \ln x\mathrm{d}x$ ***M1***

Note: The ***M1*** is for a difference of integrals. Condone absence of limits here.

attempting to use integration by parts to find $\int \ln x\mathrm{d}x$ ***(M1)***

$=\left[\frac{x^2}{2}-x\right]_1^{\mathrm{e}}-[x\ln x-x]_1^{\mathrm{e}}$ ***A1A1***

Note: Award ***A1*** for $\frac{x^2}{2}-x$ and ***A1*** for $x\ln x-x$.

Note: The second ***M1*** and second ***A1*** are independent of the first ***M1*** and the first ***A1***.

$=\frac{\mathrm{e}^2}{2}-\mathrm{e}-\frac{1}{2}\left(=\frac{\mathrm{e}^2-2\mathrm{e}-1}{2}\right)$ ***A1***

[5 marks]

e. (i) **METHOD 1**

consider for example $h(x)=x-1-\ln x$

$h(1)=0 \quad \text{and} \quad h'(x)=1-\frac{1}{x}$ ***(A1)***

as $h'(x)\geq 0 \quad \text{for} \quad x\geq 1, \quad \text{then} \quad h(x)\geq 0 \quad \text{for} \quad x\geq 1$ ***R1***

as $h'(x)\leq 0 \quad \text{for} \quad 0<x\leq 1, \quad \text{then} \quad h(x)\geq 0 \quad \text{for} \quad 0<x\leq 1$ ***R1***

so $g(x)\leq x-1,\ x\in\mathbb{R}^+$ ***AG***

METHOD 2

$g''(x)=-\frac{1}{x^2}$ ***A1***

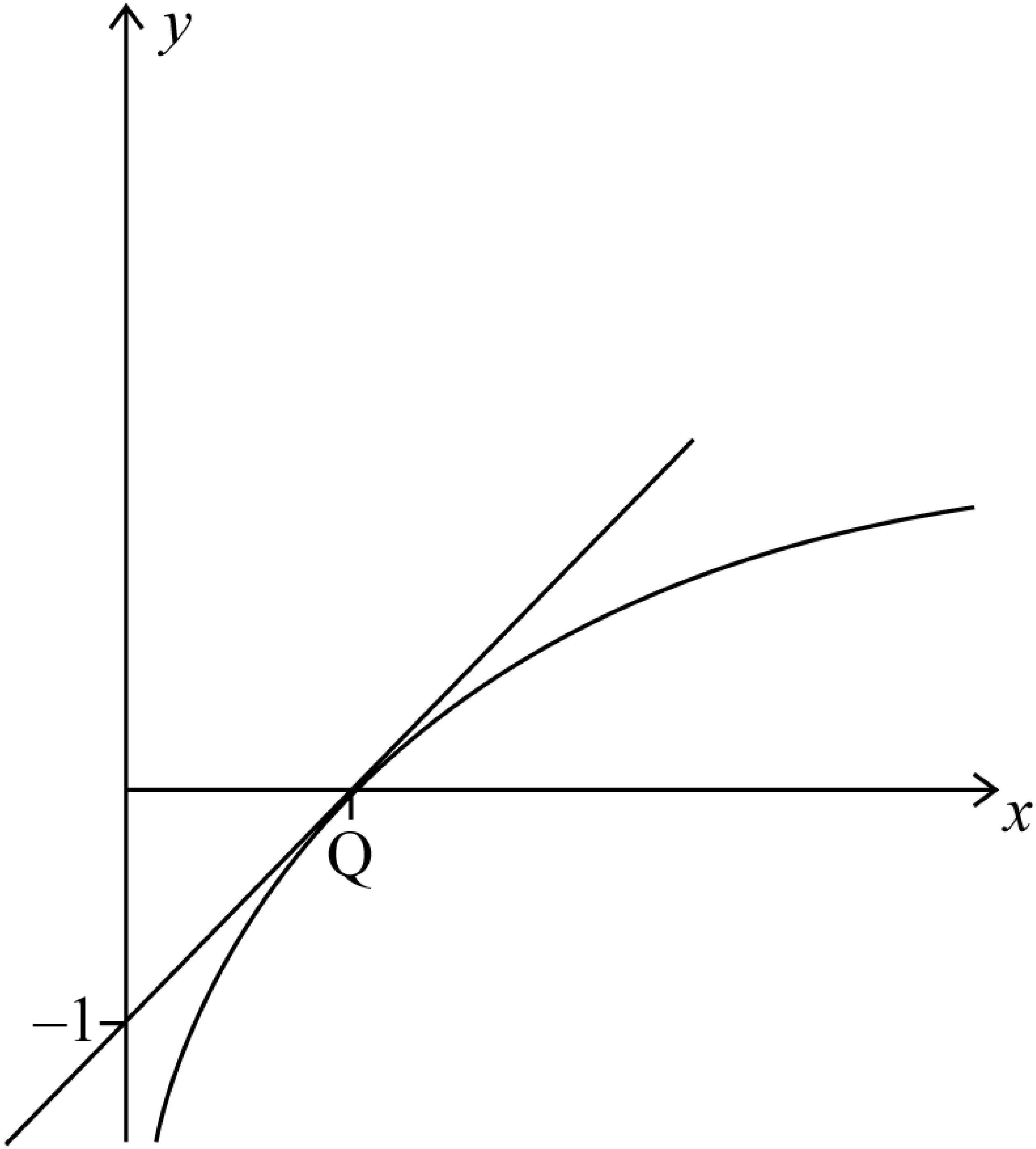
$g''(x)<0$ (concave down) for $x\in\mathbb{R}^+$ ***R1***

the graph of $y=g(x)$ is below its tangent $(y=x-1 \quad \text{at} \quad x=1)$ ***R1***

so $g(x)\leq x-1,\ x\in\mathbb{R}^+$ ***AG***

Note: The reasoning may be supported by drawn graphical arguments.

METHOD 3



clear correct graphs of $y = x - 1 \quad \text{and} \quad \ln x \quad \text{for} \quad x > 0$ ***A1A1***

statement to the effect that the graph of $\ln x$ is below the graph of its tangent at $x = 1$ ***R1AG***

(ii) replacing x by $\frac{1}{x}$ to obtain $\ln\left(\frac{1}{x}\right) \le \frac{1}{x} - 1 \left(= \frac{1-x}{x}\right)$ ***M1***

$-\ln x \le \frac{1}{x} - 1 \left(= \frac{1-x}{x}\right)$ ***(A1)***

$\ln x \ge 1 - \frac{1}{x} \left(= \frac{x-1}{x}\right)$ ***A1***

so $\frac{x-1}{x} \le g(x),\ x \in \mathbb{R}^+$ ***AG***

[6 marks]

Total [23 marks]

Examiners report

a. Generally very well done, even by candidates who had shown considerable weaknesses elsewhere on the paper.

b. Generally very well done, even by candidates who had shown considerable weaknesses elsewhere on the paper.

c. Generally very well done, even by candidates who had shown considerable weaknesses elsewhere on the paper.

d. A productive question for many candidates, but some didn't realise that a difference of areas/integrals was required.

e. (i) Many candidates adopted a graphical approach, but sometimes with unconvincing reasoning.

(ii) Poorly answered. Many candidates applied the suggested substitution only to one side of the inequality, and then had to fudge the answer.

The random variable X has probability density function f where

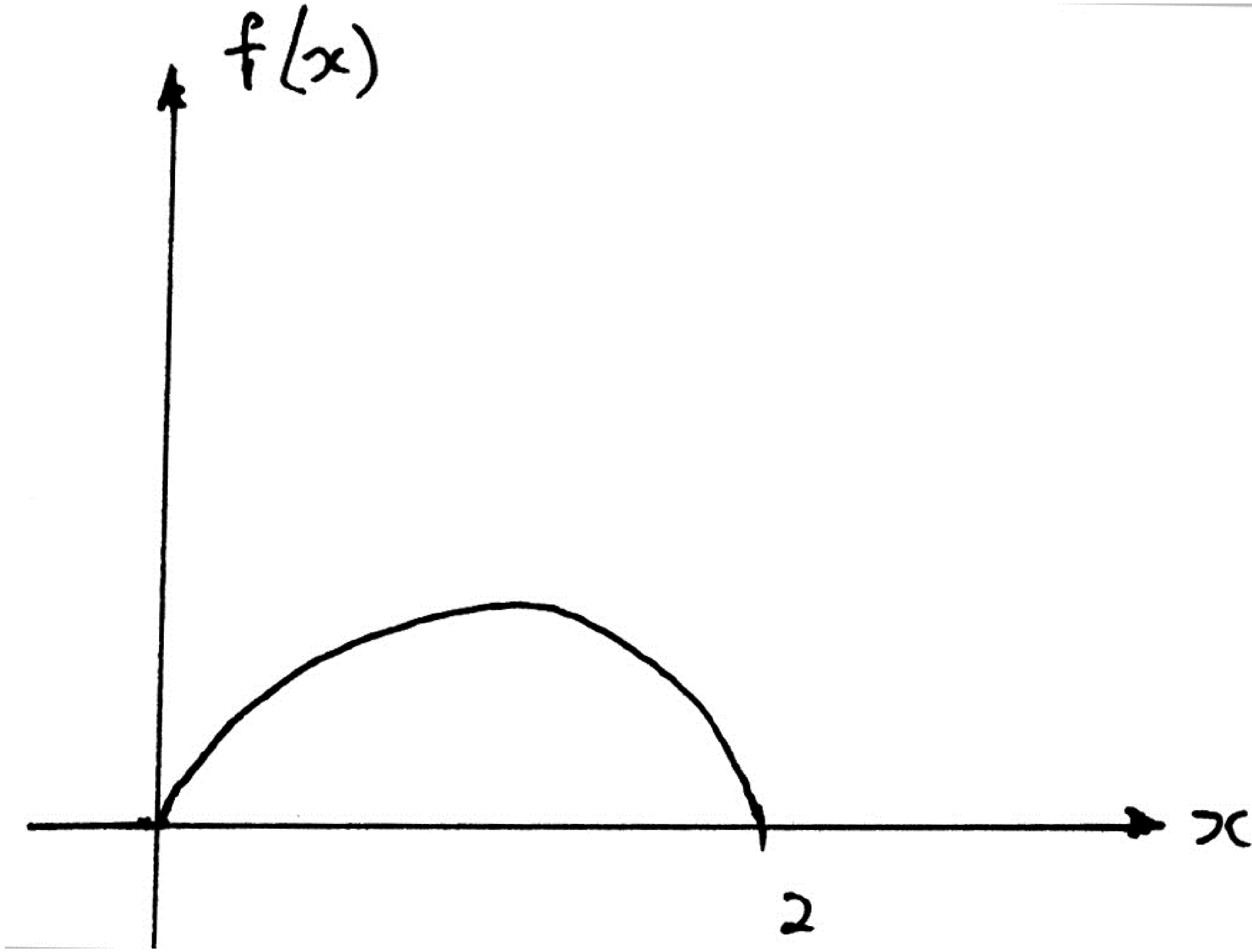
$$f(x) = \begin{cases} kx(x+1)(2-x), & 0 \leqslant x \leqslant 2 \\ 0, & \text{otherwise} . \end{cases}$$

a. Sketch the graph of the function. You are not required to find the coordinates of the maximum. [1]

b. Find the value of k . [5]

Markscheme

a.



A1

Note: Award *A1* for intercepts of 0 and 2 and a concave down curve in the given domain .

Note: Award *A0* if the cubic graph is extended outside the domain [0, 2] .

[1 mark]

b. $\int_0^2 kx(x+1)(2-x)\mathrm{d}x = 1$ ***(M1)***

Note: The correct limits and =1 must be seen but may be seen later.

$k\int_0^2 (-x^3+x^2+2x)\mathrm{d}x = 1$ ***A1***

$k\left[-\frac{1}{4}x^4+\frac{1}{3}x^3+x^2\right]_0^2 = 1$ ***M1***

$k\left(-4+\frac{8}{3}+4\right) = 1$ ***(A1)***

$k = \frac{3}{8}$ ***A1***

[5 marks]

Examiners report

a. Most candidates completed this question well. A number extended the graph beyond the given domain.

b. Most candidates completed this question well. A number extended the graph beyond the given domain.

Let $f\left(x\right) = \frac{2-3x^5}{2x^3},\ x \in \mathbb{R},\ x \ne 0.$

a. The graph of $y = f\left(x\right)$ has a local maximum at A. Find the coordinates of A. [5]

b.i. Show that there is exactly one point of inflexion, B, on the graph of $y = f\left(x\right)$. [5]

b.ii. The coordinates of B can be expressed in the form B$\left(2^a,\ b \times 2^{-3a}\right)$ where *a*, *b*$\in \mathbb{Q}$. Find the value of *a* and the value of *b*. [3]

c. Sketch the graph of $y = f\left(x\right)$ showing clearly the position of the points A and B. [4]

Markscheme

a. attempt to differentiate ***(M1)***

$f'\left(x\right) = -3x^{-4} - 3x$ ***A1***

Note: Award ***M1*** for using quotient or product rule award ***A1*** if correct derivative seen even in unsimplified form, for example $f'\left(x\right) = \frac{-15x^4 \times 2x^3 - 6x^2\left(2-3x^5\right)}{\left(2x^3\right)^2}$.

$-\frac{3}{x^4} - 3x = 0$ ***M1***

$\Rightarrow x^5 = -1 \Rightarrow x = -1$ ***A1***

$\mathrm{A}\left(-1,\ -\frac{5}{2}\right)$ ***A1***

[5 marks]

b.i. $f''\left(x\right) = 0$ ***M1***

$f''\left(x\right) = 12x^{-5} - 3\left(=0\right)$ ***A1***

Note: Award ***A1*** for correct derivative seen even if not simplified.

$\Rightarrow x = \sqrt[5]{4}\left(=2^{\frac{2}{5}}\right)$ ***A1***

hence (at most) one point of inflexion ***R1***

Note: This mark is independent of the two ***A1*** marks above. If they have shown or stated their equation has only one solution this mark can be awarded.

$f''\left(x\right)$ changes sign at $x = \sqrt[5]{4}\left(=2^{\frac{2}{5}}\right)$ ***R1***

so exactly one point of inflexion

[5 marks]

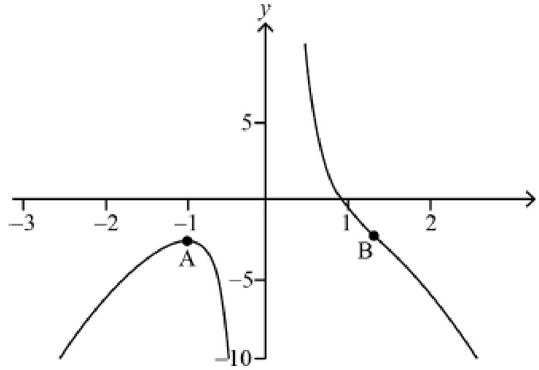
b.ii. $x = \sqrt[5]{4} = 2^{\frac{2}{5}}\left(\Rightarrow a = \frac{2}{5}\right)$ ***A1***

$f\left(2^{\frac{2}{5}}\right) = \frac{2-3\times 2^2}{2\times 2^{\frac{6}{5}}} = -5 \times 2^{-\frac{6}{5}}\left(\Rightarrow b = -5\right)$ ***(M1)A1***

Note: Award ***M1*** for the substitution of their value for x into $f\left(x\right)$.

[3 marks]

c.



A1A1A1A1

A1 for shape for $x < 0$
A1 for shape for $x > 0$
A1 for maximum at A
A1 for POI at B.

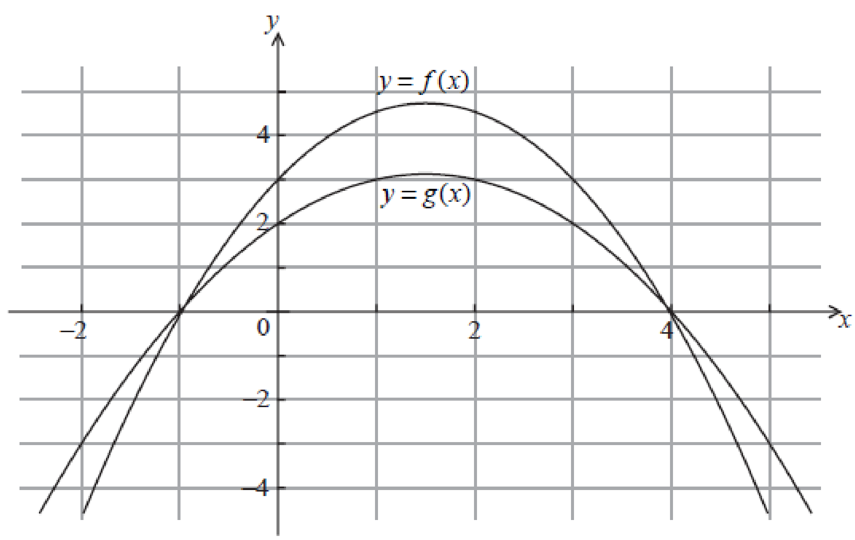
Note: Only award last two ***A1***s if A and B are placed in the correct quadrants, allowing for follow through.

[4 marks]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]
c. [N/A]

Shown below are the graphs of $y = f(x)$ and $y = g(x)$.



If $(f \circ g)(x) = 3$, find all possible values of x.

Markscheme

$g(x) = 0$ or 3 ***(M1)(A1)***

$x = -1$ or 4 or 1 or 2 ***A1A1***

Notes: Award ***A1A1*** for all four correct values,

A1A0 for two or three correct values,

A0A0 for less than two correct values.

Award ***M1*** and corresponding ***A*** marks for correct attempt to find expressions for *f* and *g*.

[4 marks]

Examiners report

A small number of candidates gave correct and well explained answers. Many candidates answered the question without showing any kind of work and in many cases it was clear that candidates were guessing and clearly did not know about composition of functions. A number of candidates attempted to find expressions for both functions but made little progress and wasted time.

The function *f* is defined on the domain $x \geqslant 0$ by $f(x) = \mathrm{e}^x - x^{\mathrm{e}}$.

a. (i) Find an expression for $f'(x)$. [3]

(ii) Given that the equation $f'(x) = 0$ has two roots, state their values.

b. Sketch the graph of *f* , showing clearly the coordinates of the maximum and minimum. [3]

c. Hence show that $\mathrm{e}^{\pi} > \pi^{\mathrm{e}}$. [1]

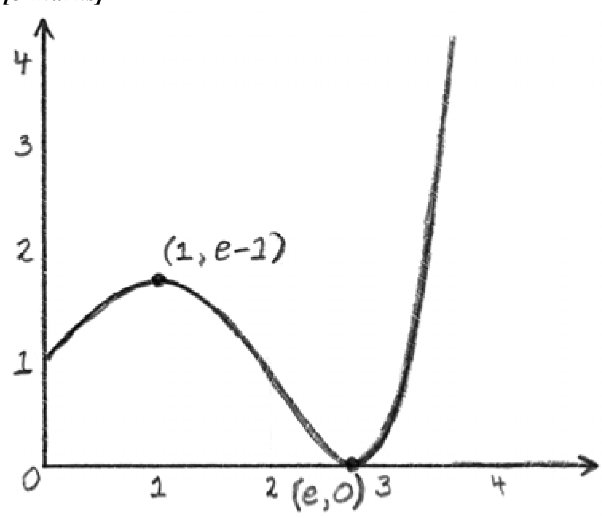
Markscheme

a. (i) $f'(x) = \mathrm{e}^x - \mathrm{e}x^{\mathrm{e}-1}$ ***A1***

(ii) by inspection the two roots are 1, e ***A1A1***

[3 marks]

b.



A3

Note: Award ***A1*** for maximum, ***A1*** for minimum and ***A1*** for general shape.

[3 marks]

c. from the graph: $\mathrm{e}^x > x^\mathrm{e}$ for all $x > 0$ except $x = \mathrm{e}$ ***R1***

putting $x = \pi$, conclude that $\mathrm{e}^\pi > \pi^\mathrm{e}$ ***AG***

[1 mark]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

The function f is defined on the domain $\left[0,\ \frac{3\pi}{2}\right]$ by $f(x) = e^{-x}\cos x$.

a. State the two zeros of f . [1]

b. Sketch the graph of f . [1]

c. The region bounded by the graph, the x-axis and the y-axis is denoted by A and the region bounded by the graph and the x-axis is denoted by B . Show that the ratio of the area of A to the area of B is [7]

$$\frac{e^\pi\left(e^{\frac{\pi}{2}}+1\right)}{e^\pi+1}.$$

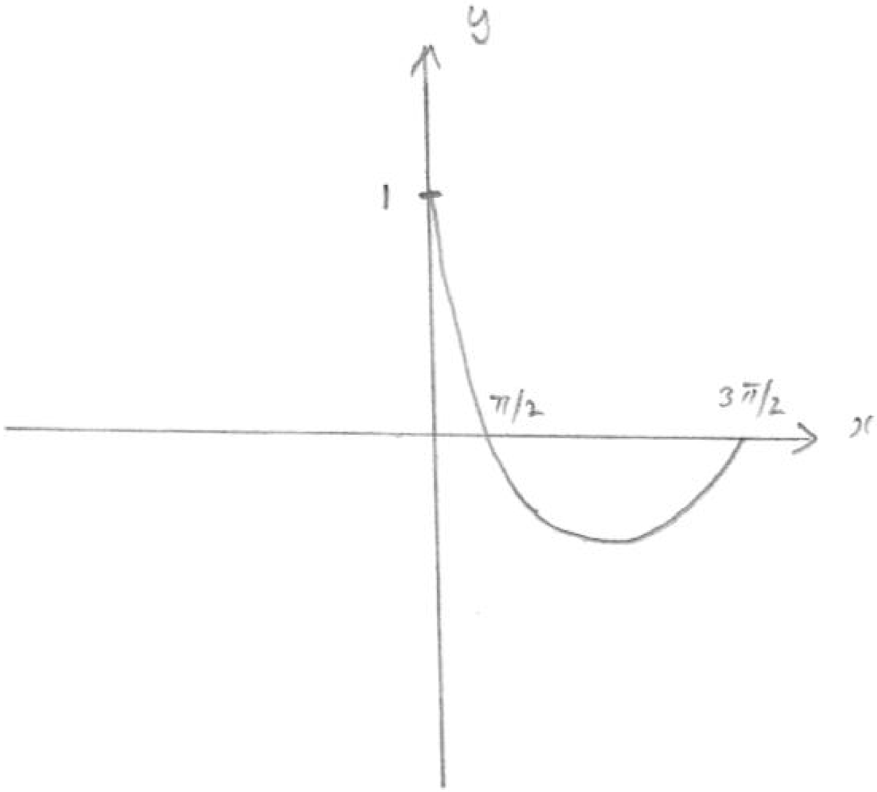
Markscheme

a. $e^{-x}\cos x = 0$

$\Rightarrow x = \frac{\pi}{2},\ \frac{3\pi}{2}$ ***A1***

[1 mark]

b.



A1

Note: Accept any form of concavity for $x \in \left[0, \frac{\pi}{2}\right]$.

Note: Do not penalize unmarked zeros if given in part (a).

Note: Zeros written on diagram can be used to allow the mark in part (a) to be awarded retrospectively.

[1 mark]

c. attempt at integration by parts ***M1***

EITHER

$I = \int e^{-x} \cos x dx = -e^{-x} \cos x dx - \int e^{-x} \sin x dx$ ***A1***

$\Rightarrow I = -e^{-x} \cos x dx - \left[-e^{-x} \sin x + \int e^{-x} \cos x dx\right]$ ***A1***

$\Rightarrow I = \frac{e^{-x}}{2}(\sin x - \cos x) + C$ ***A1***

Note: Do not penalize absence of *C*.

OR

$I = \int e^{-x} \cos x dx = e^{-x} \sin x + \int e^{-x} \sin x dx$ ***A1***

$I = e^{-x} \sin x - e^{-x} \cos x - \int e^{-x} \cos x dx$ ***A1***

$\Rightarrow I = \frac{e^{-x}}{2}(\sin x - \cos x) + C$ ***A1***

Note: Do not penalize absence of *C*.

THEN

$\int_0^{\frac{\pi}{2}} e^{-x} \cos x dx = \left[\frac{e^{-x}}{2}(\sin x - \cos x)\right]_0^{\frac{\pi}{2}} = \frac{e^{-\frac{\pi}{2}}}{2} + \frac{1}{2}$ ***A1***

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-x} \cos x dx = \left[\frac{e^{-x}}{2}(\sin x - \cos x)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -\frac{e^{-\frac{3\pi}{2}}}{2} - \frac{e^{-\frac{\pi}{2}}}{2}$ ***A1***

ratio of *A*:*B* is $\frac{\frac{e^{-\frac{\pi}{2}}}{2} + \frac{1}{2}}{\frac{e^{-\frac{3\pi}{2}}}{2} + \frac{e^{-\frac{\pi}{2}}}{2}}$

$= \frac{e^{\frac{3\pi}{2}}\left(e^{-\frac{\pi}{2}} + 1\right)}{e^{\frac{3\pi}{2}}\left(e^{-\frac{3\pi}{2}} + e^{-\frac{\pi}{2}}\right)}$ ***M1***

$= \frac{e^{\pi}\left(e^{\frac{\pi}{2}} + 1\right)}{e^{\pi} + 1}$ ***AG***

[7 marks]

Examiners report

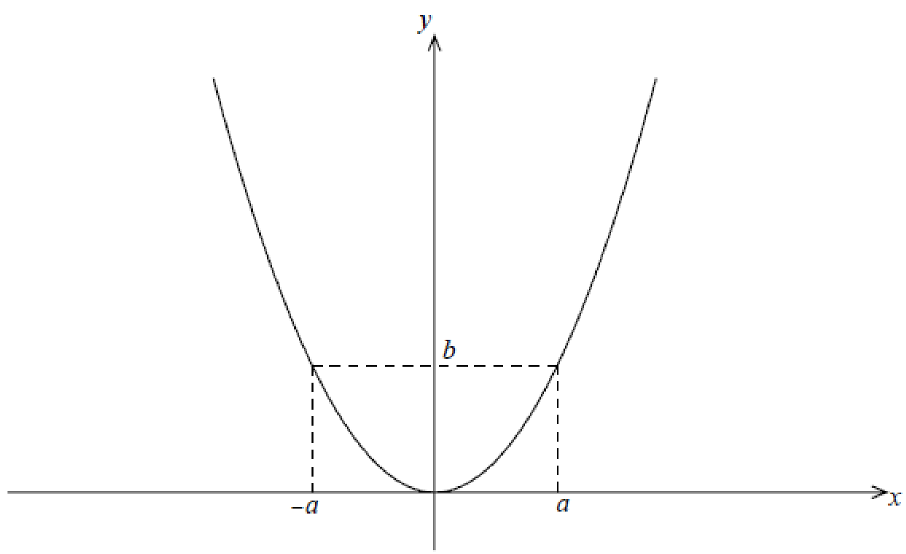
a. Many candidates stated the two zeros of *f* correctly but the graph of *f* was often incorrectly drawn. In (c), many candidates failed to realise that integration by parts had to be used twice here and even those who did that often made algebraic errors, usually due to the frequent changes of sign.

b. Many candidates stated the two zeros of *f* correctly but the graph of *f* was often incorrectly drawn. In (c), many candidates failed to realise that integration by parts had to be used twice here and even those who did that often made algebraic errors, usually due to the frequent changes of sign.

c. Many candidates stated the two zeros of *f* correctly but the graph of *f* was often incorrectly drawn. In (c), many candidates failed to realise that integration by parts had to be used twice here and even those who did that often made algebraic errors, usually due to the frequent changes of sign.

The diagram below shows the graph of the function $y = f(x)$, defined for all $x \in \mathbb{R}$,

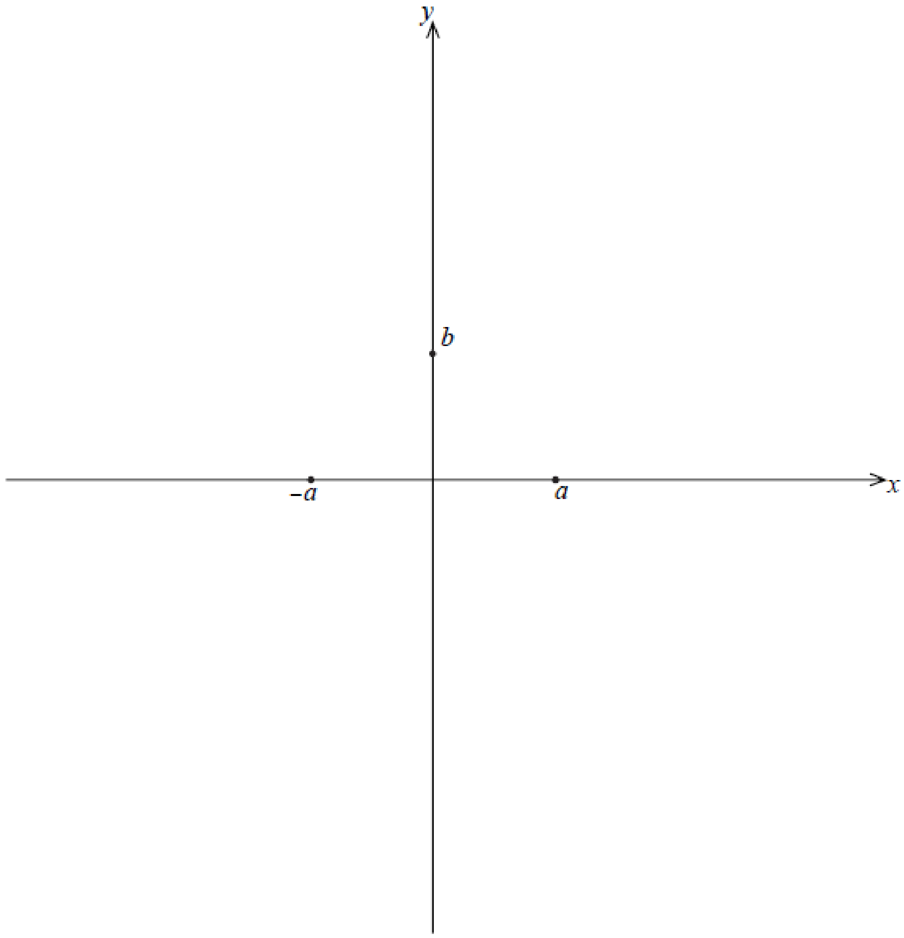
where $b > a > 0$.



Consider the function $g(x) = \frac{1}{f(x-a)-b}$.

a. Find the largest possible domain of the function g . [2]

b. On the axes below, sketch the graph of $y = g(x)$. On the graph, indicate any asymptotes and local maxima or minima, and write down their equations and coordinates. [6]



Markscheme

a. $f(x-a) \neq b$ ***(M1)***

$x \neq 0$ and $x \neq 2a$ (or equivalent) ***A1***

[2 marks]

b. vertical asymptotes $x = 0$, $x = 2a$ ***A1***

horizontal asymptote $y = 0$ ***A1***

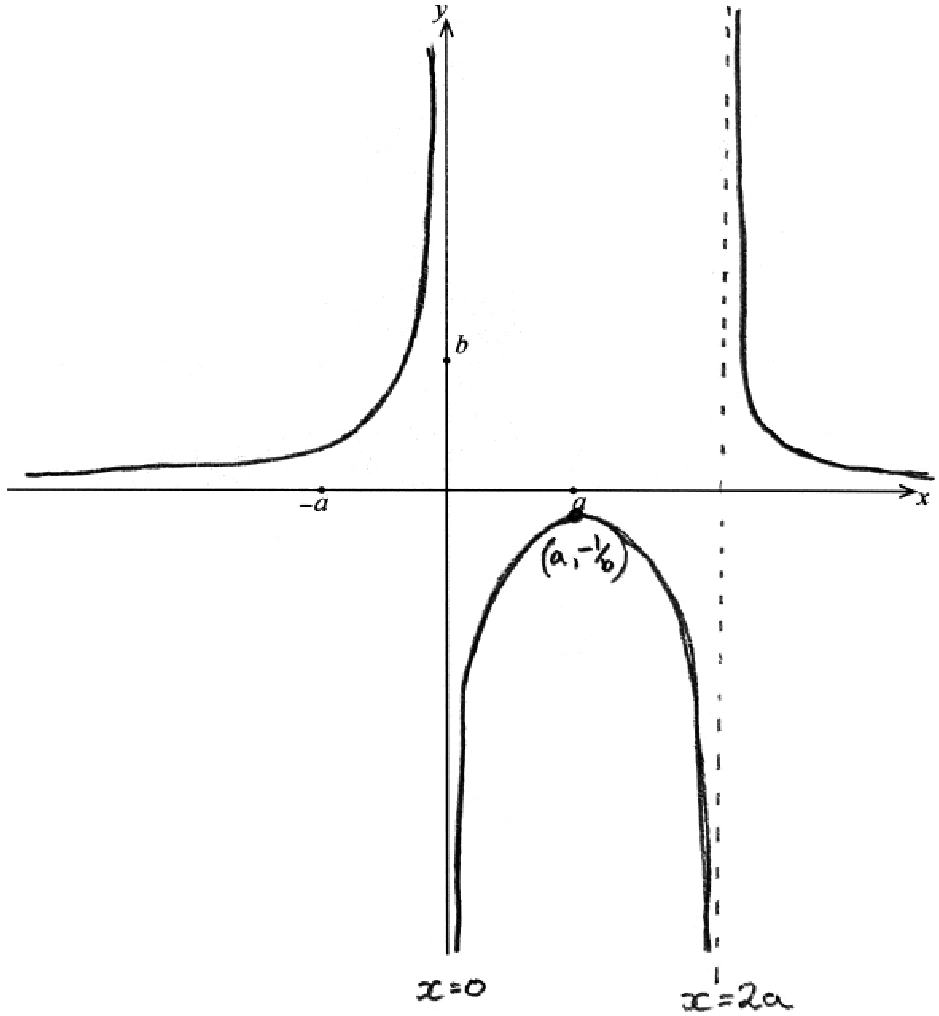
Note: Equations must be seen to award these marks.

maximum $\left(a, -\frac{1}{b}\right)$ ***A1A1***

Note: Award ***A1*** for correct *x*-coordinate and ***A1*** for correct *y*-coordinate.

one branch correct shape ***A1***

other 2 branches correct shape ***A1***



[6 marks]

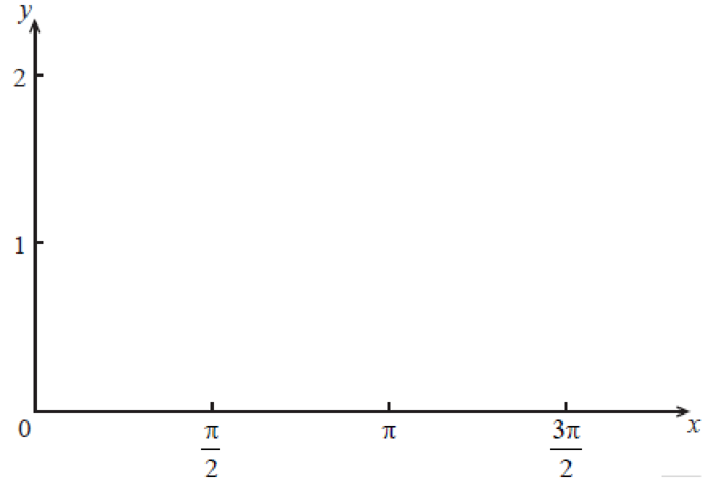
Examiners report

a. A significant number of candidates did not answer this question. Among the candidates who attempted it there were many who had difficulties in connecting vertical asymptotes and the domain of the function and dealing with transformations of graphs. In a few cases candidates managed to answer (a) but provided an answer to (b) which was inconsistent with the domain found.

b. A significant number of candidates did not answer this question. Among the candidates who attempted it there were many who had difficulties in connecting vertical asymptotes and the domain of the function and dealing with transformations of graphs. In a few cases candidates managed to answer (a) but provided an answer to (b) which was inconsistent with the domain found.

Given that $f(x) = 1 + \sin x,\ 0 \leqslant x \leqslant \frac{3\pi}{2}$,

a. sketch the graph of f; [1]

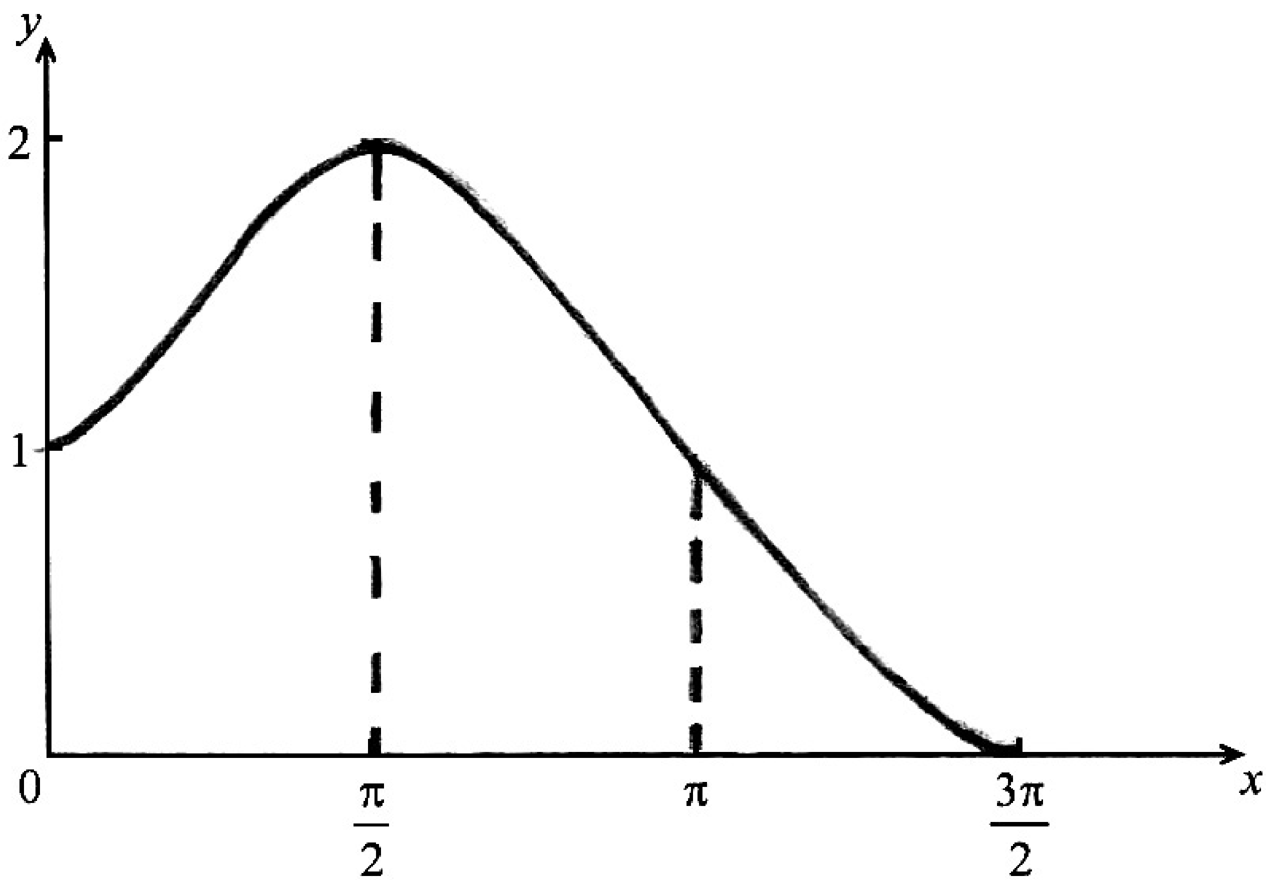


b. show that $(f(x))^2 = \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$; [1]

c. find the volume of the solid formed when the graph of f is rotated through 2π radians about the x-axis. [4]

Markscheme

a.



A1

[1 mark]

b. $(1+\sin x)^2 = 1 + 2\sin x + \sin^2 x$

$= 1 + 2\sin x + \frac{1}{2}(1 - \cos 2x)$ ***A1***

$= \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$ ***AG***

[1 mark]

c. $V = \pi\int_0^{\frac{3\pi}{2}} (1+\sin x)^2 \mathrm{d}x$ ***(M1)***

$= \pi\int_0^{\frac{3\pi}{2}} \left(\frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x\right) \mathrm{d}x$

$= \pi\left[\frac{3}{2}x - 2\cos x - \frac{\sin 2x}{4}\right]_0^{\frac{3\pi}{2}}$ ***A1***

$= \frac{9\pi^2}{4} + 2\pi$ ***A1A1***

[4 marks]

Examiners report

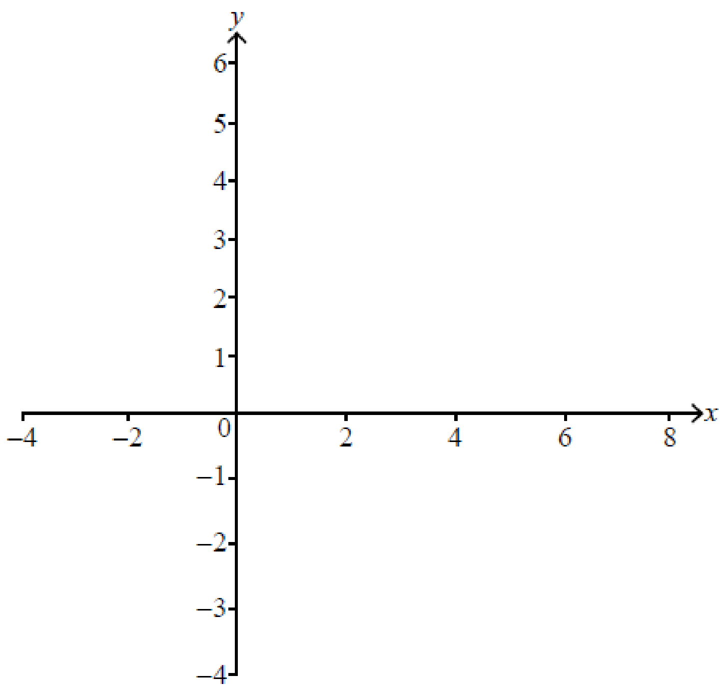
a. Parts (a) and (b) were almost invariably correctly answered by candidates. In (c), most errors involved the integration of $\cos(2x)$ and the insertion of the limits.

b. Parts (a) and (b) were almost invariably correctly answered by candidates. In (c), most errors involved the integration of $\cos(2x)$ and the insertion of the limits.

c. Parts (a) and (b) were almost invariably correctly answered by candidates. In (c), most errors involved the integration of $\cos(2x)$ and the insertion of the limits.

---

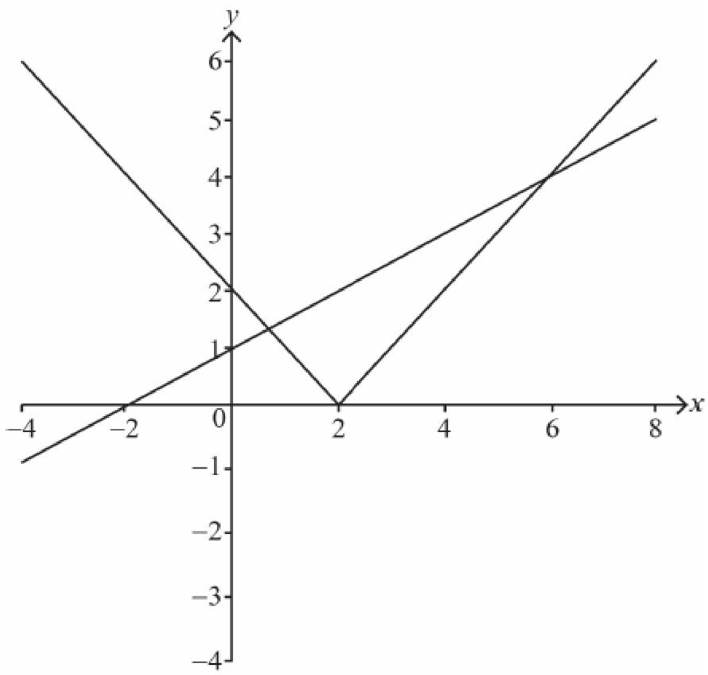
a. Sketch the graphs of $y = \frac{x}{2} + 1$ and $y = |x - 2|$ on the following axes. [3]



b. Solve the equation $\frac{x}{2} + 1 = |x - 2|$. [4]

Markscheme

a.



straight line graph with correct axis intercepts ***A1***

modulus graph: V shape in upper half plane ***A1***

modulus graph having correct vertex and *y*-intercept ***A1***

[3 marks]

b. **METHOD 1**

attempt to solve $\frac{x}{2}+1=x-2$ ***(M1)***

$x=6$ ***A1***

Note: Accept $x=6$ using the graph.

attempt to solve (algebraically) $\frac{x}{2}+1=2-x$ ***M1***

$x=\frac{2}{3}$ ***A1***

[4 marks]

METHOD 2

$\left(\frac{x}{2}+1\right)^2=(x-2)^2$ ***M1***

$\frac{x^2}{4}+x+1=x^2-4x+4$

$0=\frac{3x^2}{4}-5x+3$

$3x^2-20x+12=0$

attempt to factorise (or equivalent) ***M1***

$(3x-2)(x-6)=0$

$x=\frac{2}{3}$ ***A1***

$x=6$ ***A1***

[4 marks]

Examiners report

a. [N/A]
b. [N/A]

A function is defined as $f(x) = k\sqrt{x}$, with $k > 0$ and $x \geqslant 0$.

(a) Sketch the graph of $y = f(x)$.

(b) Show that f is a one-to-one function.

(c) Find the inverse function, $f^{-1}(x)$ and state its domain.

(d) If the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at the point (4, 4) find the value of k .

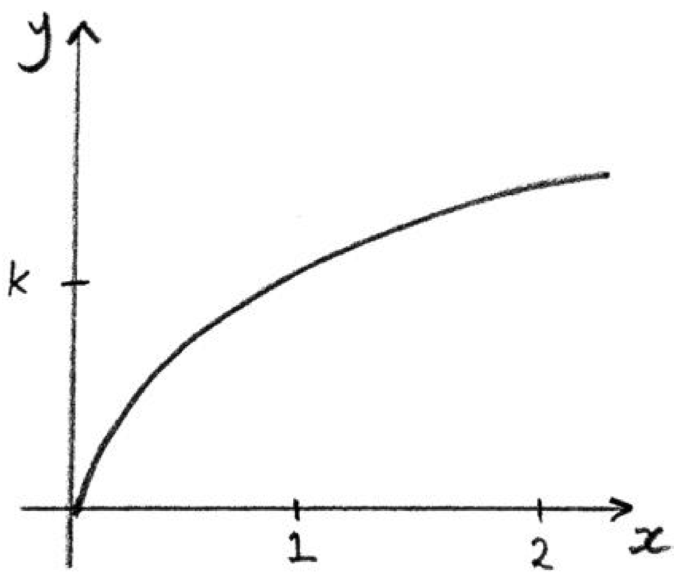
(e) Consider the graphs of $y = f(x)$ and $y = f^{-1}(x)$ using the value of k found in part (d).

(i) Find the area enclosed by the two graphs.

(ii) The line $x = c$ cuts the graphs of $y = f(x)$ and $y = f^{-1}(x)$ at the points P and Q respectively. Given that the tangent to $y = f(x)$ at point P is parallel to the tangent to $y = f^{-1}(x)$ at point Q find the value of c .

Markscheme

(a)



A1

Note: Award ***A1*** for correct concavity, passing through (0, 0) and increasing.

Scales need not be there.

[1 mark]

(b) a statement involving the application of the Horizontal Line Test or equivalent ***A1***

[1 mark]

(c) $y = k\sqrt{x}$

for either $x = k\sqrt{y}$ or $x = \frac{y^2}{k^2}$ ***A1***

$f^{-1}(x) = \frac{x^2}{k^2}$ ***A1***

$\operatorname{dom}\left(f^{-1}(x)\right)=[0,\infty[$ ***A1***

[3 marks]

(d) $\frac{x^2}{k^2}=k\sqrt{x}$ or equivalent method ***M1***

$k=\sqrt{x}$

$k=2$ ***A1***

[2 marks]

(e) (i) $A=\int_a^b(y_1-y_2)\mathrm{d}x$ ***(M1)***

$A=\int_0^4\left(2x^{\frac{1}{2}}-\frac{1}{4}x^2\right)\mathrm{d}x$ ***A1***

$=\left[\frac{4}{3}x^{\frac{3}{2}}-\frac{1}{12}x^3\right]_0^4$ ***A1***

$=\frac{16}{3}$ ***A1***

(ii) attempt to find either $f'(x)$ or $(f^{-1})'(x)$ ***M1***

$f'(x)=\frac{1}{\sqrt{x}},\ \left((f^{-1})'(x)=\frac{x}{2}\right)$ ***A1A1***

$\frac{1}{\sqrt{c}}=\frac{c}{2}$ ***M1***

$c=2^{\frac{2}{3}}$ ***A1***

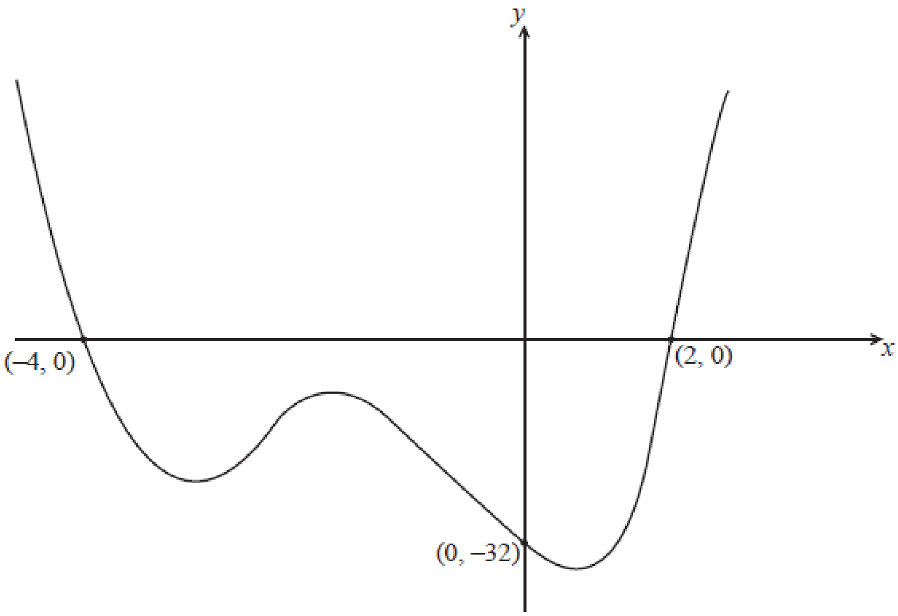
[9 marks]

Total [16 marks]

Examiners report

Many students could not sketch the function. There was confusion between the vertical and horizontal line test for one-to-one functions. A significant number of students gave long and inaccurate explanations for a one-to-one function. Finding the inverse was done very well by most students although the notation used was generally poor. The domain of the inverse was ignored by many or done incorrectly even if the sketch was correct. Many did not make the connections between the parts of the question. An example of this was the number of students who spent time finding the point of intersection in part e) even though it was given in d).

The graph of a polynomial function *f* of degree 4 is shown below.



A.a Given that $(x+\mathrm{i}y)^2=-5+12\mathrm{i},\ x,\ y\in\mathbb{R}$. Show that [2]

(i) $x^2-y^2=-5$;
(ii) $xy=6$.

A.b Hence find the two square roots of $-5+12\mathrm{i}$. [5]

A.c For any complex number z , show that $(z^*)^2=(z^2)^*$. [3]

A.d Hence write down the two square roots of $-5-12\mathrm{i}$. [2]

B.a Explain why, of the four roots of the equation $f(x)=0$, two are real and two are complex. [2]

B.b The curve passes through the point $(-1,\ -18)$. Find $f(x)$ in the form [5]

$f(x)=(x-a)(x-b)(x^2+cx+d),$ where $a,\ b,\ c,\ d\in\mathbb{Z}$.

B.c Find the two complex roots of the equation $f(x)=0$ in Cartesian form. [2]

B.d Draw the four roots on the complex plane (the Argand diagram). [2]

B.e Express each of the four roots of the equation in the form $re^{\mathrm{i}\theta}$. [6]

Markscheme

A.a (i) $(x+\mathrm{i}y)^2=-5+12\mathrm{i}$

$x^2+2\mathrm{i}xy+\mathrm{i}^2y^2=-5+12\mathrm{i}$ ***A1***
(ii) equating real and imaginary parts ***M1***
$x^2-y^2=-5$ ***AG***
$xy=6$ ***AG***
[2 marks]

A.b substituting ***M1***

EITHER
$x^2-\frac{36}{x^2}=-5$
$x^4+5x^2-36=0$ ***A1***
$x^2=4,\ -9$ ***A1***
$x=\pm2$ and $y=\pm3$ ***(A1)***
OR
$\frac{36}{y^2}-y^2=-5$
$y^4-5y^2-36=0$ ***A1***
$y^2=9,\ -4$ ***A1***
$y^2=\pm3$ **and** $x=\pm2$ ***(A1)***
Note: Accept solution by inspection if completely correct.

THEN
the square roots are $(2 + 3\mathrm{i})$ and $(-2 - 3\mathrm{i})$ ***A1***
[5 marks]

A.c **EITHER**

consider $z = x + \mathrm{i}y$
$z^* = x - \mathrm{i}y$
$(z^*)^2 = x^2 - y^2 - 2\mathrm{i}xy$ ***A1***
$(z^2) = x^2 - y^2 + 2\mathrm{i}xy$ ***A1***
$(z^2)^* = x^2 - y^2 - 2\mathrm{i}xy$ ***A1***
$(z^*)^2 = (z^2)^*$ ***AG***
OR
$z^* = r\mathrm{e}^{-\mathrm{i}\theta}$
$(z^*)^2 = r^2\mathrm{e}^{-2\mathrm{i}\theta}$ ***A1***
$z^2 = r^2\mathrm{e}^{2\mathrm{i}\theta}$ ***A1***
$(z^2)^* = r^2\mathrm{e}^{-2\mathrm{i}\theta}$ ***A1***
$(z^*)^2 = (z^2)^*$ ***AG***
[3 marks]

A.d $(2 - 3\mathrm{i})$ and $(-2 + 3\mathrm{i})$ ***A1A1***

[2 marks]

B.a the graph crosses the *x*-axis twice, indicating two real roots ***R1***

since the quartic equation has four roots and only two are real, the other two roots must be complex ***R1***
[2 marks]

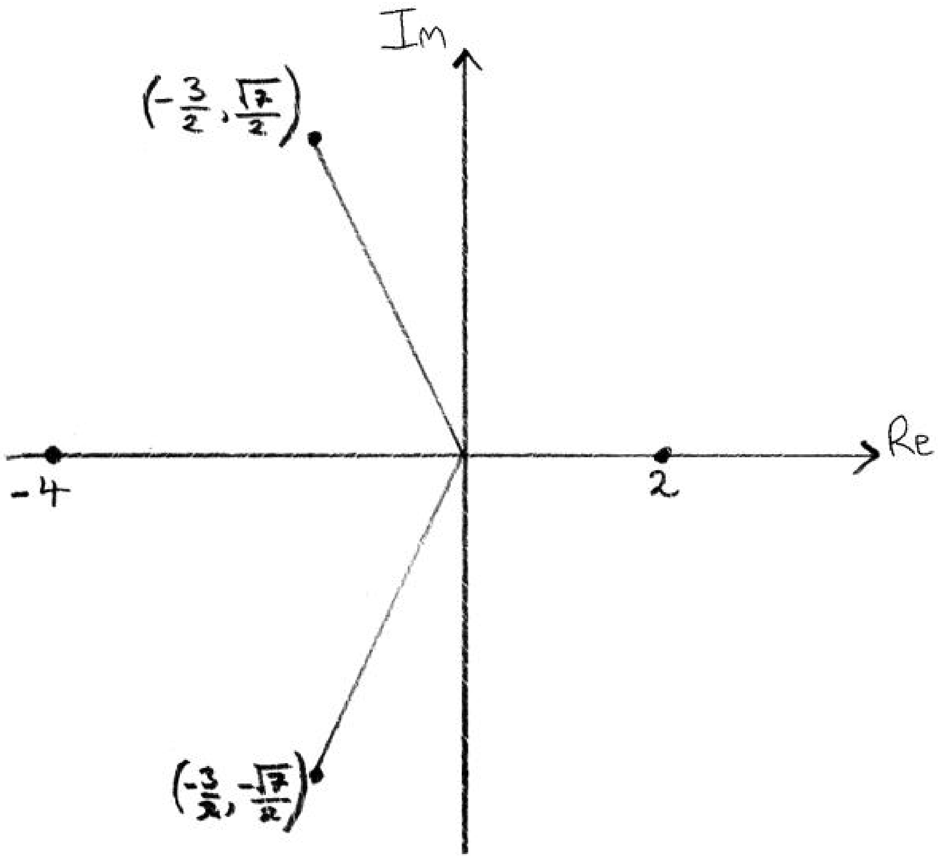
B.b $f(x) = (x + 4)(x - 2)(x^2 + cx + d)$ ***A1A1***

$f(0) = -32 \Rightarrow d = 4$ ***A1***
Since the curve passes through $(-1,\ -18)$,
$-18 = 3 \times (-3)(5 - c)$ ***M1***
$c = 3$ ***A1***
Hence $f(x) = (x + 4)(x - 2)(x^2 + 3x + 4)$
[5 marks]

B.c $x = \frac{-3 \pm \sqrt{9-16}}{2}$ ***(M1)***

$\Rightarrow x = -\frac{3}{2} \pm \mathrm{i}\frac{\sqrt{7}}{2}$ ***A1***
[2 marks]

B.d.



A1A1

Note: Accept points or vectors on complex plane.
Award ***A1*** for two real roots and ***A1*** for two complex roots.

[2 marks]

B.e. real roots are $4\mathrm{e}^{\mathrm{i}\pi}$ and $2\mathrm{e}^{\mathrm{i}0}$ ***A1A1***

considering $-\frac{3}{2} \pm \mathrm{i}\frac{\sqrt{7}}{2}$

$r = \sqrt{\frac{9}{4} + \frac{7}{4}} = 2$ ***A1***

finding θ using $\arctan\left(\frac{\sqrt{7}}{3}\right)$ ***M1***

$\theta = \arctan\left(\frac{\sqrt{7}}{3}\right) + \pi$ or $\theta = \arctan\left(-\frac{\sqrt{7}}{3}\right) + \pi$ ***A1***

$\Rightarrow z = 2\mathrm{e}^{\mathrm{i}\left(\arctan\left(\frac{\sqrt{7}}{3}\right)+\pi\right)}$ or $\Rightarrow z = 2\mathrm{e}^{\mathrm{i}\left(\arctan\left(\frac{-\sqrt{7}}{3}\right)+\pi\right)}$ ***A1***

Note: Accept arguments in the range $-\pi$ to π or 0 to 2π .
Accept answers in degrees.

[6 marks]

Examiners report

A.a. Since (a) was a ‘show that’ question, it was essential for candidates to give a convincing explanation of how the quoted results were obtained.

Many candidates just wrote

$$(x + \mathrm{i}y)^2 = x^2 - y^2 + 2\mathrm{i}xy = -5 + 12\mathrm{i}$$

$$\text{Therefore } x^2 - y^2 = -5 \text{ and } xy = 6$$

This was not given full credit since it simply repeated what was given in the question. Candidates were expected to make it clear that they were equating real and imaginary parts. In (b), candidates who attempted to use de Moivre’s Theorem to find the square roots were given no credit since the question stated ‘hence’.

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B.a In (a), the explanations were often unconvincing. Candidates were expected to make it clear that the two intersections with the *x*-axis gave two real roots and, since the polynomial was a quartic and therefore had four zeros, the other two roots must be complex. Candidates who made vague statements such as ‘the graph shows two real roots’ were not given full credit. In (b), most candidates stated the values of *a* and *b* correctly but algebraic errors often led to incorrect values for the other parameters. Candidates who failed to solve (b) correctly were unable to solve (c), (d) and (e) correctly although follow through was used where possible.

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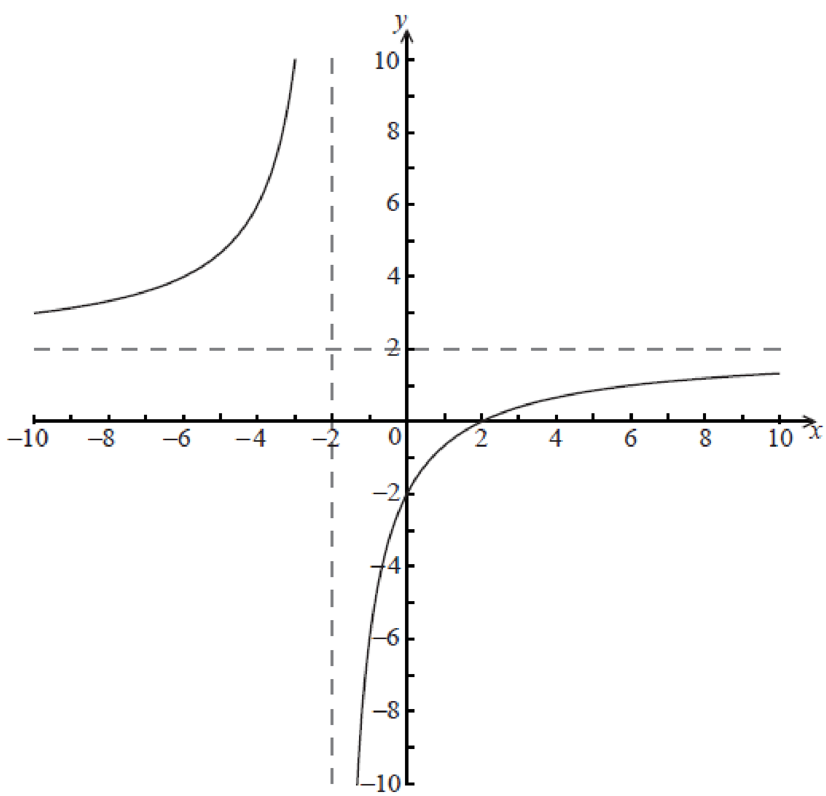
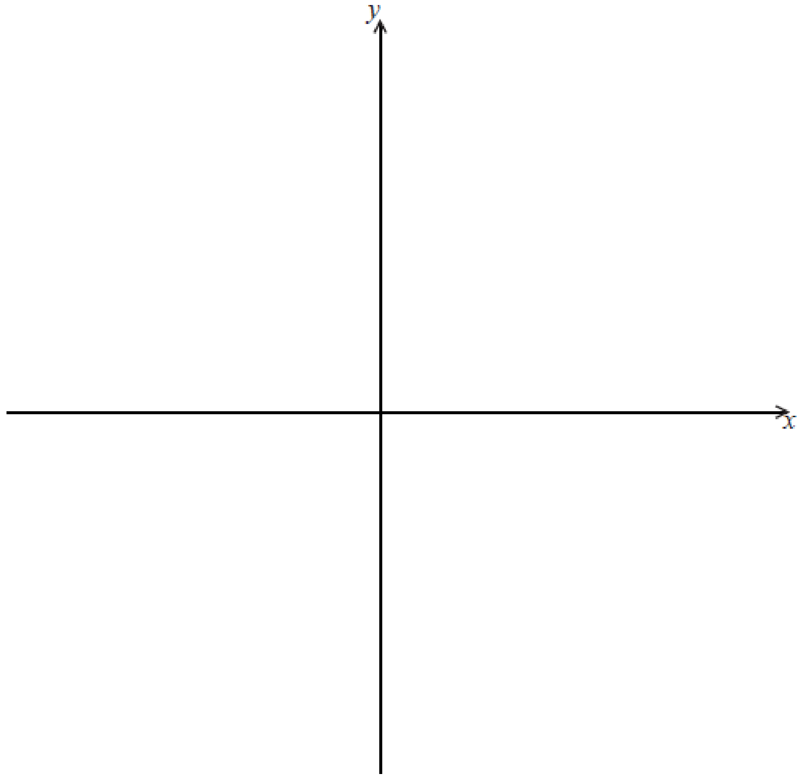
The graph of $y = \frac{a+x}{b+cx}$ is drawn below.



(a) Find the value of a, the value of b and the value of c.

(b) Using the values of a, b and c found in part (a), sketch the graph of $y = \left|\frac{b+cx}{a+x}\right|$ on the axes below, showing clearly all intercepts and asymptotes.

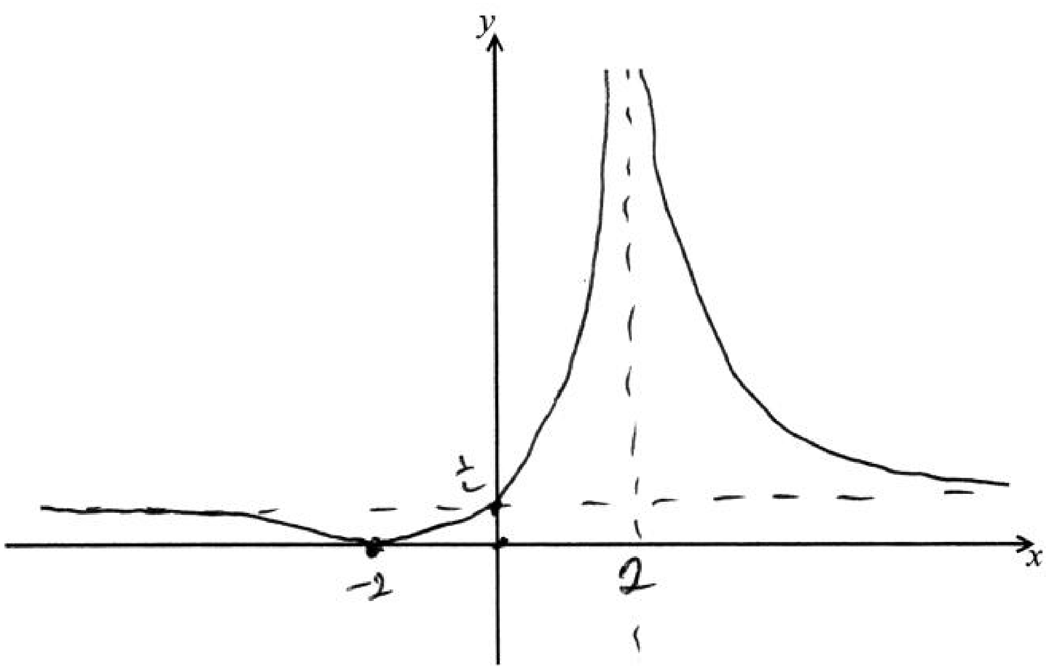


Markscheme

(a) an attempt to use either asymptotes or intercepts ***(M1)***

$a = -2,\ b = 1,\ c = \frac{1}{2}$ ***A1A1A1***

(b)



A4

Note: Award ***A1*** for both asymptotes,

A1 for both intercepts,

A1, ***A1*** for the shape of each branch, ignoring shape at $(x = -2)$.

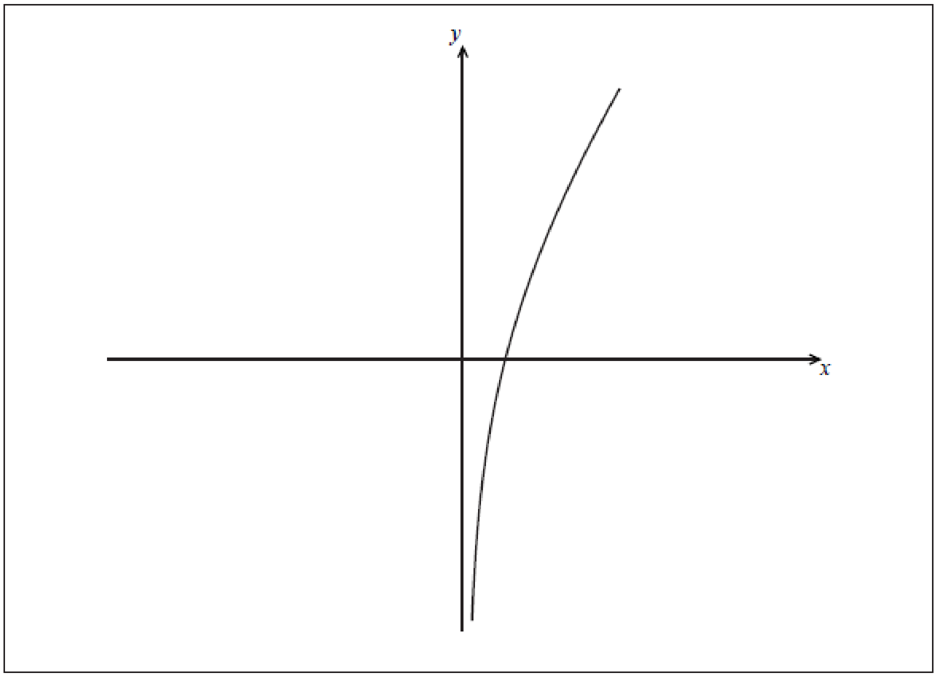
[8 marks]

Examiners report

It was pleasing to see a lot of good work with part (a), though some candidates lost marks due to problems with the algebra which led to one or more incorrect values. Regarding part (b), most candidates did not succeed in finding the new intercepts and asymptotes and were unable to apply the absolute value function. A significant number of candidates misread part (b) and took it as the modulus of the graph in part (a).

The graph below shows $y = f(x)$, where $f(x) = x + \ln x$.

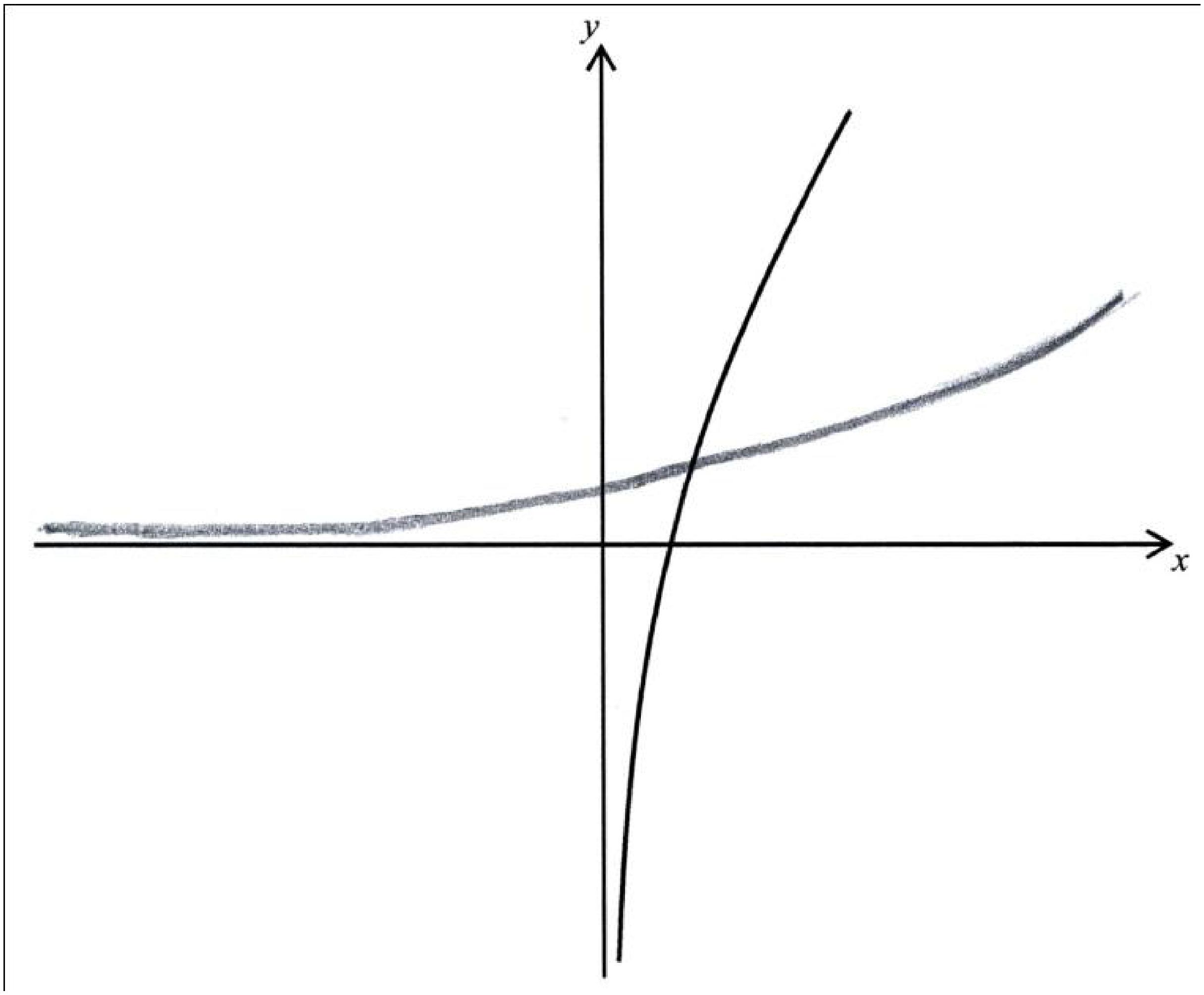
(a) On the graph below, sketch the curve $y = f^{-1}(x)$.



(b) Find the coordinates of the point of intersection of the graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$.

Markscheme

(a)



A1A1

Note: Award ***A1*** for correct asymptote with correct behaviour and ***A1*** for shape.
[2 marks]

(b) intersect on $y = x$ ***(M1)***
$x + \ln x = x \Rightarrow \ln x = 0$ ***(A1)***
intersect at (1, 1) ***A1*** ***A1***
[4 marks]

Total [6 marks]

Examiners report

Most students were able to sketch the correct graph, but then many failed to recognise that they could use their solution to determine the solution of part (b). Those who did were generally successful and those who embarked on attempts to find the inverse function did not realise that this was leading them nowhere.

Consider the function $f(x) = \frac{\ln x}{x}$, $0 < x < \mathrm{e}^2$.

a. (i) Solve the equation $f'(x) = 0$. [5]

(ii) Hence show the graph of f has a local maximum.

(iii) Write down the range of the function f .

b. Show that there is a point of inflexion on the graph and determine its coordinates. [5]

c. Sketch the graph of $y = f(x)$, indicating clearly the asymptote, x-intercept and the local maximum. [3]

d. Now consider the functions $g(x) = \frac{\ln|x|}{x}$ and $h(x) = \frac{\ln|x|}{|x|}$, where $0 < x < e^2$. [6]

(i) Sketch the graph of $y = g(x)$.

(ii) Write down the range of g .

(iii) Find the values of x such that $h(x) > g(x)$.

Markscheme

a. (i) $f'(x) = \frac{x\frac{1}{x} - \ln x}{x^2}$ ***M1A1***

$= \frac{1 - \ln x}{x^2}$

so $f'(x) = 0$ when $\ln x = 1$, *i.e.* $x = \mathrm{e}$ ***A1***

(ii) $f'(x) > 0$ when $x < \mathrm{e}$ and $f'(x) < 0$ when $x > \mathrm{e}$ ***R1***

hence local maximum ***AG***

Note: Accept argument using correct second derivative.

(iii) $y \leqslant \frac{1}{\mathrm{e}}$ ***A1***

[5 marks]

b. $f''(x) = \frac{x^2\frac{-1}{x} - (1 - \ln x)2x}{x^4}$ ***M1***

$= \frac{-x - 2x + 2x\ln x}{x^4}$

$= \frac{-3 + 2\ln x}{x^3}$ ***A1***

Note: May be seen in part (a).

$f''(x) = 0$ ***(M1)***

$-3 + 2\ln x = 0$

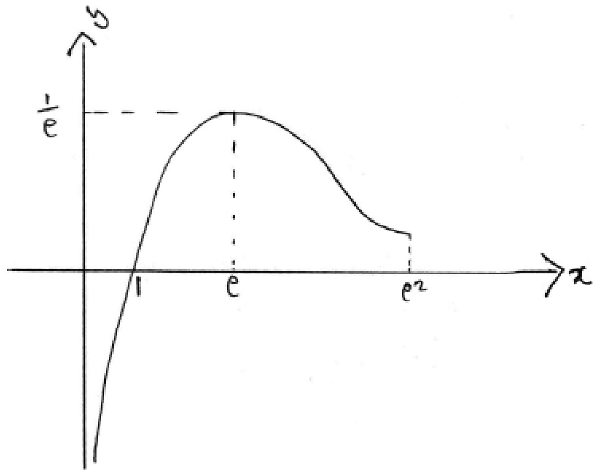
$x = \mathrm{e}^{\frac{3}{2}}$

since $f''(x) < 0$ when $x < \mathrm{e}^{\frac{3}{2}}$ and $f''(x) > 0$ when $x > \mathrm{e}^{\frac{3}{2}}$ ***R1***

then point of inflexion $\left(\mathrm{e}^{\frac{3}{2}}, \frac{3}{2\mathrm{e}^{\frac{3}{2}}}\right)$ ***A1***

[5 marks]

c.

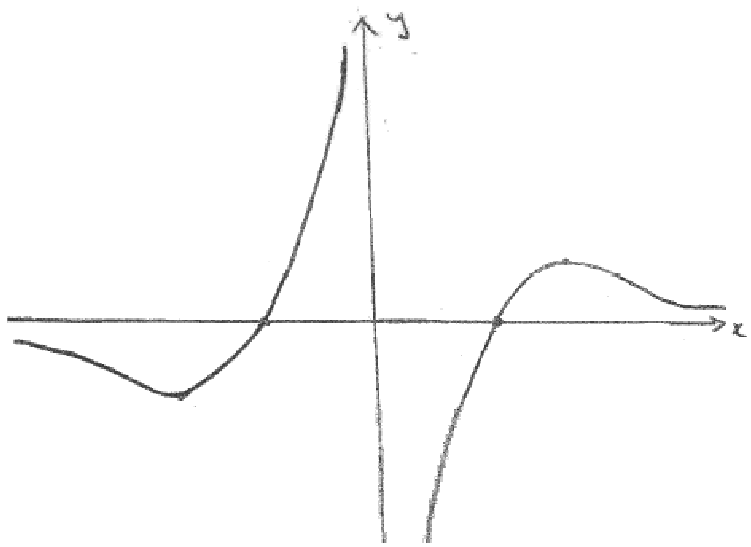


A1A1A1

Note: Award *A1* for the maximum and intercept, *A1* for a vertical asymptote and *A1* for shape (including turning concave up).

[3 marks]

d. (i)

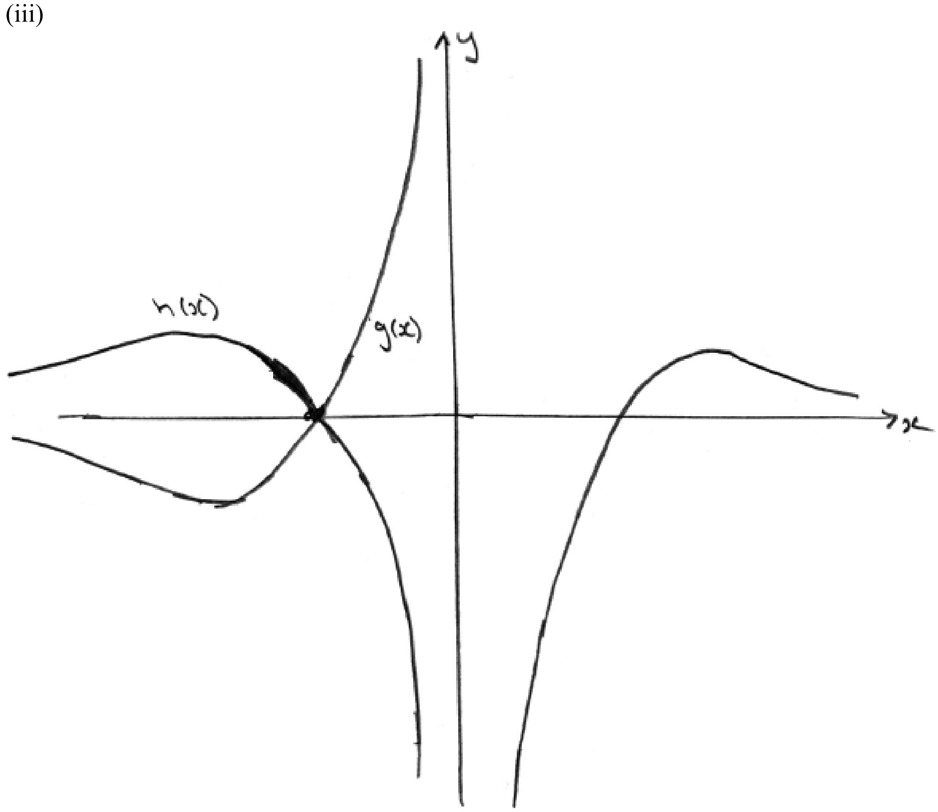


A1A1

Note: Award *A1* for each correct branch.

(ii) all real values *A1*

(iii)



(M1)(A1)

Note: Award ***(M1)(A1)*** for sketching the graph of h, ignoring any graph of g.

$-\mathrm{e}^2 < x < -1$ (accept $x < -1$)   ***A1***

[6 marks]

Examiners report

a. Most candidates attempted parts (a), (b) and (c) and scored well, although many did not gain the reasoning marks for the justification of the existence of local maximum and inflexion point. The graph sketching was poorly done. A wide selection of range shapes were seen, in some cases showing little understanding of the relation between the derivatives of the function and its graph and difficulties with transformation of graphs. In some cases candidates sketched graphs consistent with their previous calculations but failed to label them properly.

b. Most candidates attempted parts (a), (b) and (c) and scored well, although many did not gain the reasoning marks for the justification of the existence of local maximum and inflexion point. The graph sketching was poorly done. A wide selection of range shapes were seen, in some cases showing little understanding of the relation between the derivatives of the function and its graph and difficulties with transformation of graphs. In some cases candidates sketched graphs consistent with their previous calculations but failed to label them properly.

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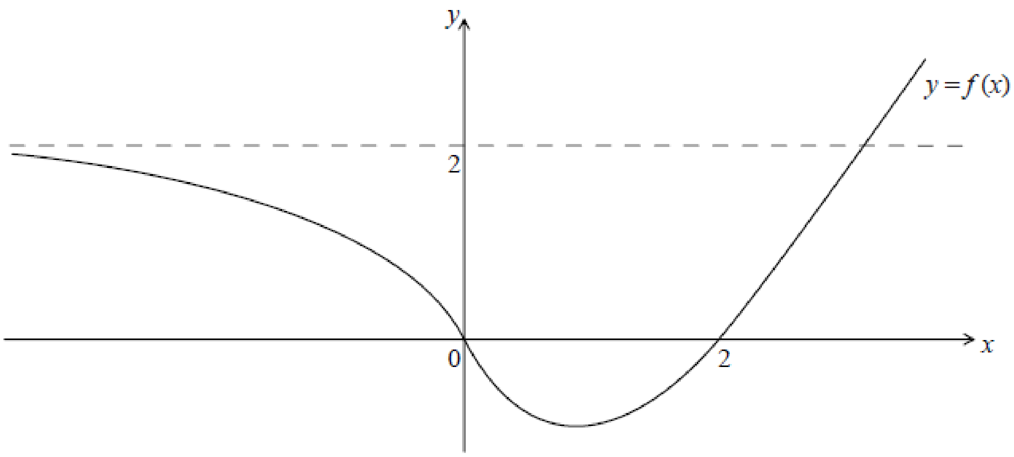
The diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = 2$.

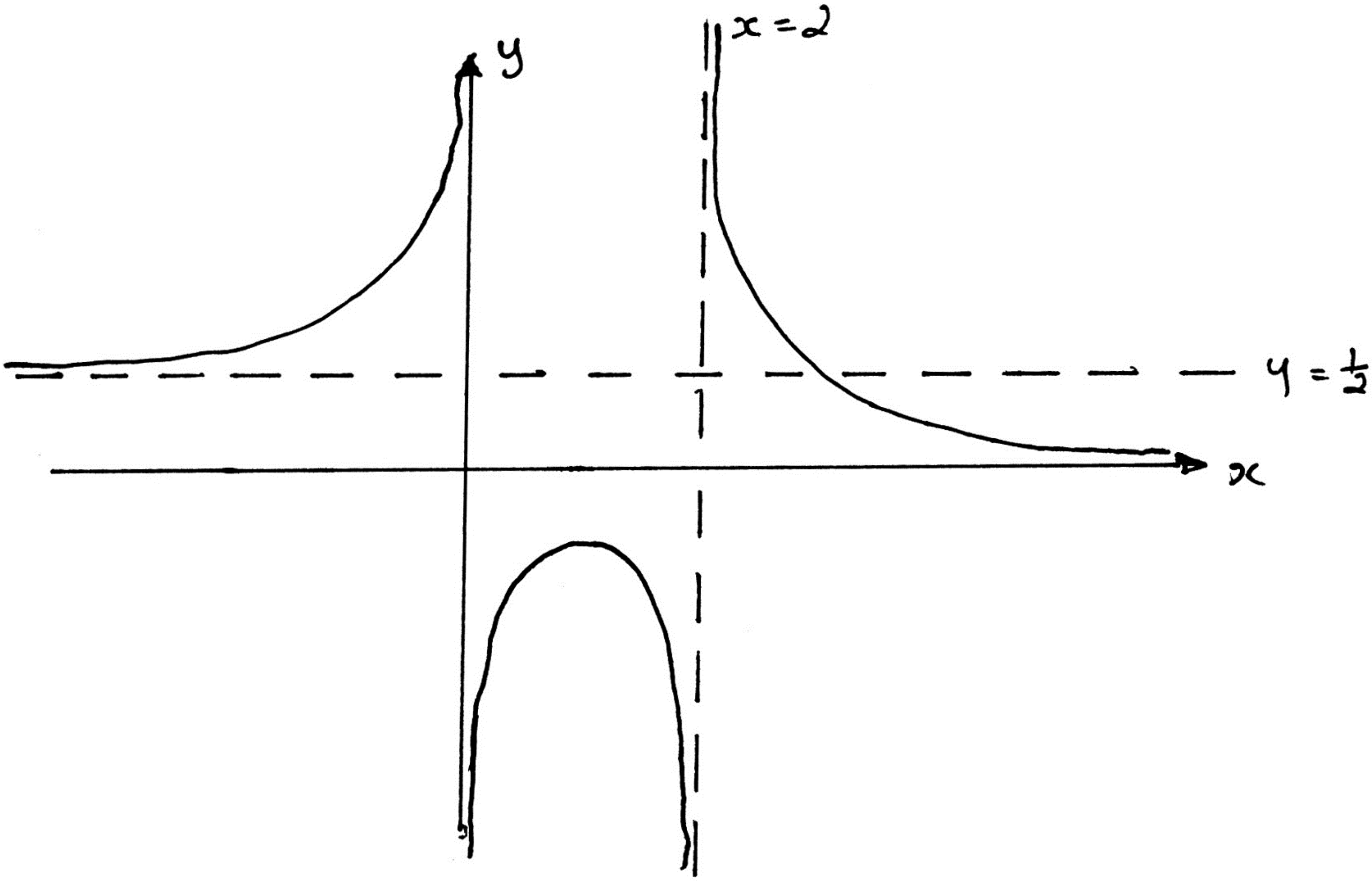


a. Sketch the graph of $y = \frac{1}{f(x)}$. [3]

b. Sketch the graph of $y = x\ f(x)$. [3]

Markscheme

a.

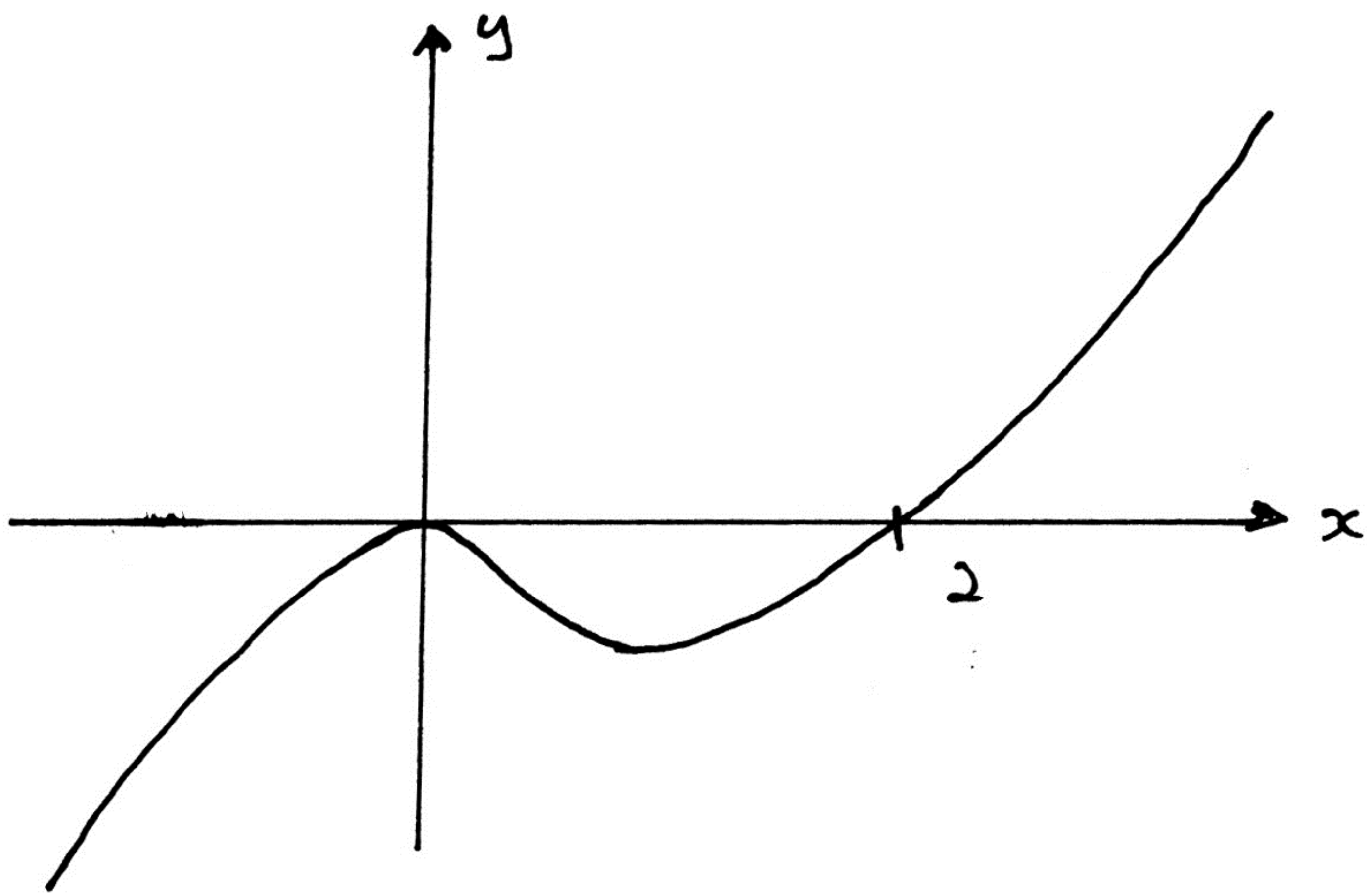


A3

Note: Award ***A1*** for each correct branch with position of asymptotes clearly indicated.
If $x = 2$ is not indicated, only penalise once.

[3 marks]

b.



A3

Note: Award ***A1*** for behaviour at $x = 0$, ***A1*** for intercept at $x = 2$, ***A1*** for behaviour for large $|x|$.

[3 marks]

Examiners report

a. Many candidates were able to find the reciprocal but many struggled with the second part. Sketches were quite poor in detail.

b. Many candidates were able to find the reciprocal but many struggled with the second part. Sketches were quite poor in detail.

---

a. (i) Sketch the graphs of $y = \sin x$ and $y = \sin 2x$, on the same set of axes, for $0 \leqslant x \leqslant \frac{\pi}{2}$. [9]

(ii) Find the x-coordinates of the points of intersection of the graphs in the domain $0 \leqslant x \leqslant \frac{\pi}{2}$.

(iii) Find the area enclosed by the graphs.

b. Find the value of $\int_0^1 \sqrt{\frac{x}{4-x}} \mathrm{d}x$ using the substitution $x = 4\sin^2\theta$. [8]

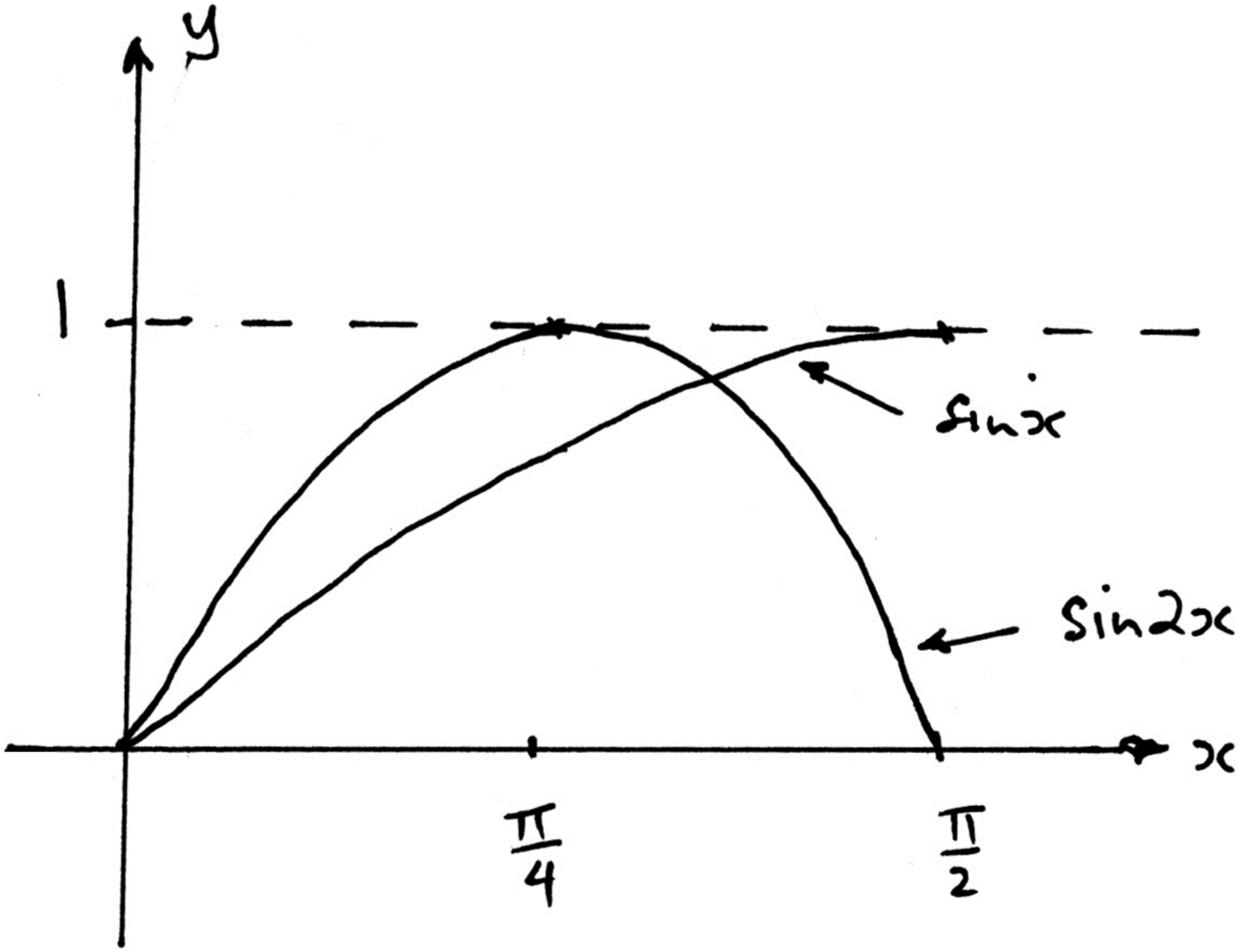
c. The increasing function *f* satisfies $f(0) = 0$ and $f(a) = b$, where $a > 0$ and $b > 0$. [8]

(i) By reference to a sketch, show that $\int_0^a f(x)\mathrm{d}x = ab - \int_0^b f^{-1}(x)\mathrm{d}x$.

(ii) **Hence** find the value of $\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x$.

Markscheme

a. (i)



A2

Note: Award *A1* for correct $\sin x$, *A1* for correct $\sin 2x$.

Note: Award *A1A0* for two correct shapes with $\frac{\pi}{2}$ and/or 1 missing.

Note: Condone graph outside the domain.

(ii) $\sin 2x = \sin x$, $0 \leqslant x \leqslant \frac{\pi}{2}$
$2\sin x\cos x - \sin x = 0$ *M1*
$\sin x(2\cos x - 1) = 0$
$x = 0, \frac{\pi}{3}$ *A1A1* *N1N1*

(iii) area $= \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)\mathrm{d}x$ *M1*
Note: Award *M1* for an integral that contains limits, not necessarily correct, with $\sin x$ and $\sin 2x$ subtracted in either order.

$= \left[-\frac{1}{2}\cos 2x + \cos x\right]_0^{\frac{\pi}{3}}$ *A1*

$= \left(-\frac{1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{3}\right) - \left(-\frac{1}{2}\cos 0 + \cos 0\right)$ ***(M1)***

$= \frac{3}{4} - \frac{1}{2}$

$= \frac{1}{4}$ ***A1***

[9 marks]

b. $\int_0^1 \sqrt{\frac{x}{4-x}}\mathrm{d}x = \int_0^{\frac{\pi}{6}} \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}} \times 8\sin\theta\cos\theta\mathrm{d}\theta$ ***M1A1A1***

Note: Award ***M1*** for substitution and reasonable attempt at finding expression for $\mathrm{d}x$ in terms of $\mathrm{d}\theta$, first ***A1*** for correct limits, second ***A1*** for correct substitution for $\mathrm{d}x$.

$\int_0^{\frac{\pi}{6}} 8\sin^2\theta\mathrm{d}\theta$ ***A1***

$\int_0^{\frac{\pi}{6}} 4 - 4\cos 2\theta\mathrm{d}\theta$ ***M1***

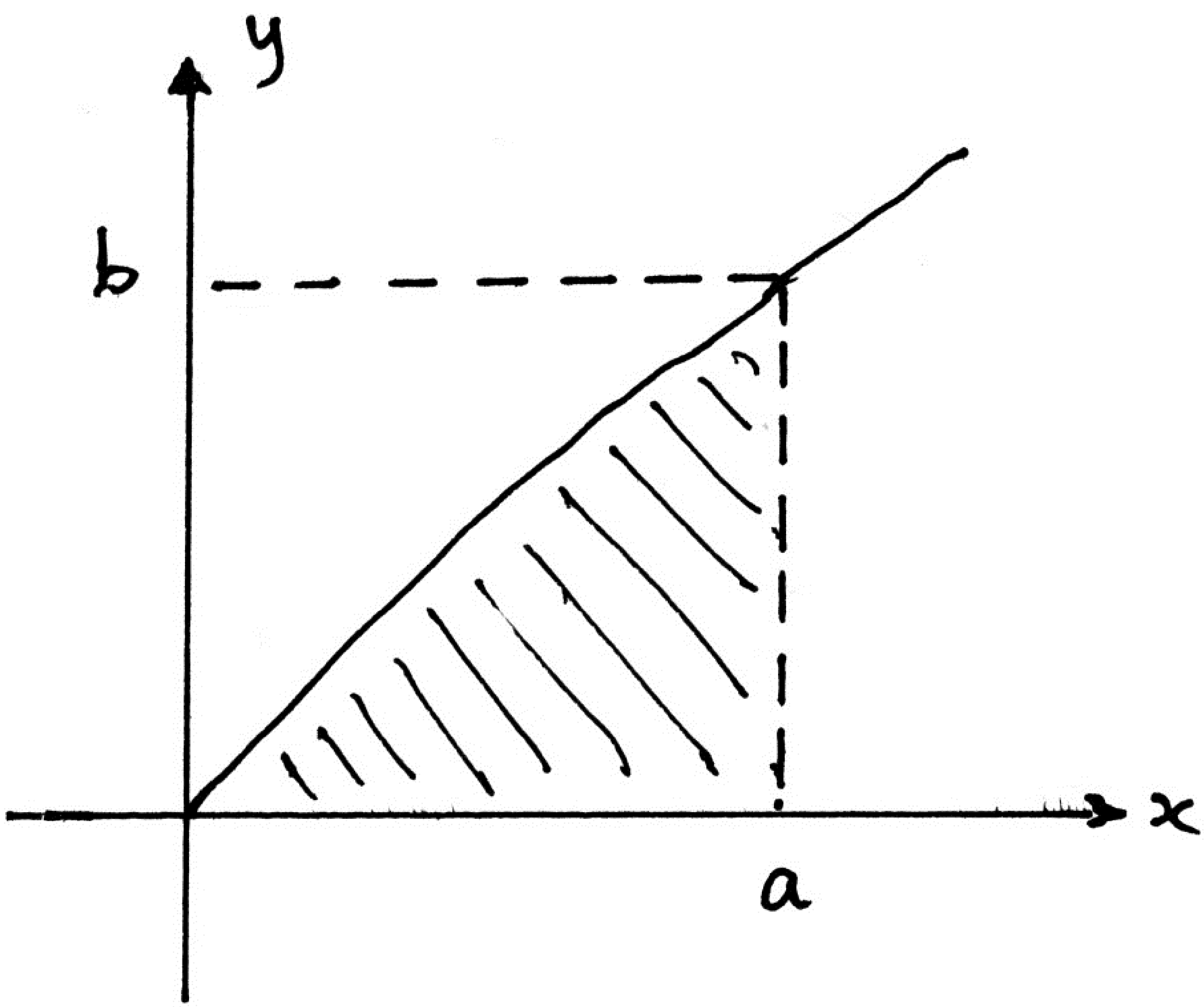
$= [4\theta - 2\sin 2\theta]_0^{\frac{\pi}{6}}$ ***A1***

$= \left(\frac{2\pi}{3} - 2\sin\frac{\pi}{3}\right) - 0$ ***(M1)***

$= \frac{2\pi}{3} - \sqrt{3}$ ***A1***

[8 marks]

c. (i)



M1

from the diagram above

the shaded area $= \int_0^a f(x)\mathrm{d}x = ab - \int_0^b f^{-1}(y)\mathrm{d}y$ ***R1***

$= ab - \int_0^b f^{-1}(x)\mathrm{d}x$ ***AG***

(ii) $f(x) = \arcsin \frac{x}{4} \Rightarrow f^{-1}(x) = 4\sin x$ ***A1***

$\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x = \frac{\pi}{3} - \int_0^{\frac{\pi}{6}} 4\sin x \mathrm{d}x$ ***M1A1A1***

Note: Award ***A1*** for the limit $\frac{\pi}{6}$ seen anywhere, ***A1*** for all else correct.

$= \frac{\pi}{3} - [-4\cos x]_0^{\frac{\pi}{6}}$ ***A1***

$= \frac{\pi}{3} - 4 + 2\sqrt{3}$ ***A1***

Note: Award no marks for methods using integration by parts.

[8 marks]

Examiners report

a. A significant number of candidates did not seem to have the time required to attempt this question satisfactorily.

Part (a) was done quite well by most but a number found sketching the functions difficult, the most common error being poor labelling of the axes.
Part (ii) was done well by most the most common error being to divide the equation by $\sin x$ and so omit the $x = 0$ value. Many recognised the value from the graph and corrected this in their final solution.
The final part was done well by many candidates.
Many candidates found (b) challenging. Few were able to substitute the *dx* expression correctly and many did not even seem to recognise the need for this term. Those that did tended to be able to find the integral correctly. Most saw the need for the double angle expression although many did not change the limits successfully.
Few candidates attempted part c). Those who did get this far managed the sketch well and were able to explain the relationship required. Among those who gave a response to this many were able to get the result although a number made errors in giving the inverse function. On the whole those who got this far did it well.

b. A significant number of candidates did not seem to have the time required to attempt this question satisfactorily.

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The final part was done well by many candidates.
Many candidates found (b) challenging. Few were able to substitute the *dx* expression correctly and many did not even seem to recognise the need for this term. Those that did tended to be able to find the integral correctly. Most saw the need for the double angle expression although many did not change the limits successfully.
Few candidates attempted part c). Those who did get this far managed the sketch well and were able to explain the relationship required. Among those who gave a response to this many were able to get the result although a number made errors in giving the inverse function. On the whole those who got this far did it well.

c. A significant number of candidates did not seem to have the time required to attempt this question satisfactorily.

Part (a) was done quite well by most but a number found sketching the functions difficult, the most common error being poor labelling of the axes.
Part (ii) was done well by most the most common error being to divide the equation by $\sin x$ and so omit the $x = 0$ value. Many recognised the value from the graph and corrected this in their final solution.
The final part was done well by many candidates.
Many candidates found (b) challenging. Few were able to substitute the *dx* expression correctly and many did not even seem to recognise the need for this term. Those that did tended to be able to find the integral correctly. Most saw the need for the double angle expression although many did not change the limits successfully.
Few candidates attempted part c). Those who did get this far managed the sketch well and were able to explain the relationship required. Among those who gave a response to this many were able to get the result although a number made errors in giving the inverse function. On the whole those who got this far did it well.

The diagram below shows a solid with volume V, obtained from a cube with edge $a > 1$ when a smaller cube with edge $\frac{1}{a}$ is removed.

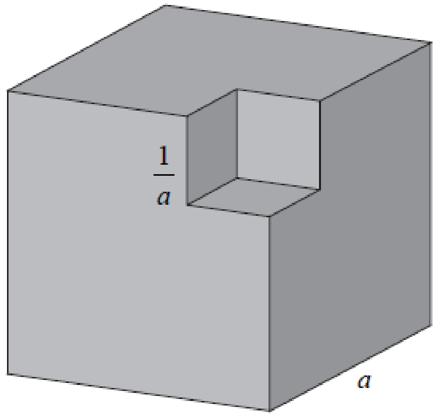


diagram not to scale

Let $x = a - \frac{1}{a}$

(a) Find V in terms of x.

(b) Hence or otherwise, show that the only value of a for which $V = 4x$ is $a = \frac{1+\sqrt{5}}{2}$.

Markscheme

(a) **METHOD 1**

$V = a^3 - \frac{1}{a^3}$ ***A1***

$x^3 = \left(a - \frac{1}{a}\right)^3$ ***M1***

$= a^3 - 3a + \frac{3}{a} - \frac{1}{a^3}$

$= a^3 - \frac{1}{a^3} - 3\left(a - \frac{1}{a}\right)$ (or equivalent) ***(A1)***

$\Rightarrow a^3 - \frac{1}{a^3} = x^3 + 3x$

$V = x^3 + 3x$ ***A1 N0***

METHOD 2

$V = a^3 - \frac{1}{a^3}$ ***A1***

attempt to use difference of cubes formula, $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$ ***M1***

$V = \left(a - \frac{1}{a}\right)\left(a^2 + 1 + \left(\frac{1}{a}\right)^2\right)$

$= \left(a - \frac{1}{a}\right)\left(\left(a - \frac{1}{a}\right)^2 + 3\right)$ ***(A1)***

$= x(x^2 + 3)$ or $x^3 + 3x$ ***A1 N0***

METHOD 3

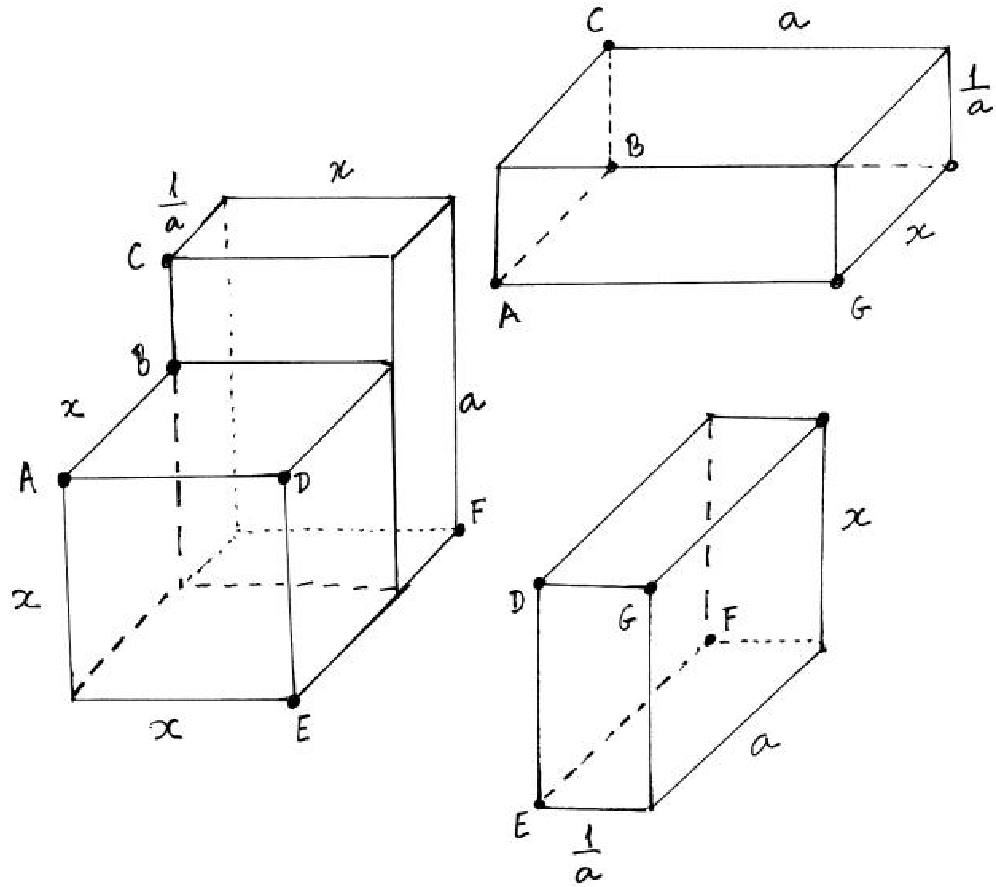


diagram showing that the solid can be decomposed ***M1***

into three congruent $x \times a \times \frac{1}{a}$ cuboids with volume x ***A1***

and a cube with edge x with volume x^3 ***A1***

so, $V = x^3 + 3x$ ***A1*** ***N0***

(b)

Note: Do not accept any method where candidate substitutes the given value of a into $x = a - \frac{1}{a}$.

METHOD 1

$V = 4x \Leftrightarrow x^3 + 3x = 4x \Leftrightarrow x^3 - x = 0$ ***M1***

$\Leftrightarrow x(x-1)(x+1) = 0$

$\Rightarrow x = 1$ as $x > 0$ ***A1***

so, $a - \frac{1}{a} = 1 \Rightarrow a^2 - a - 1 = 0 \Rightarrow a = \frac{1 \pm \sqrt{5}}{2}$ ***M1A1***

as $a > 1$, $a = \frac{1+\sqrt{5}}{2}$ ***AG*** ***N0***

METHOD 2

$a^3 - \frac{1}{a^3} = 4\left(a - \frac{1}{a}\right) \Rightarrow a^6 - 4a^4 + 4a^2 - 1 = 0 \Leftrightarrow (a^2-1)(a^4 - 3a^2 + 1) = 0$ ***M1A1***

as $a > 1 \Rightarrow a^2 > 1$, $a^2 = \frac{3+\sqrt{5}}{2} \Leftrightarrow a^2 = \sqrt{\left(\frac{1+\sqrt{5}}{2}\right)^2}$ ***M1A1***

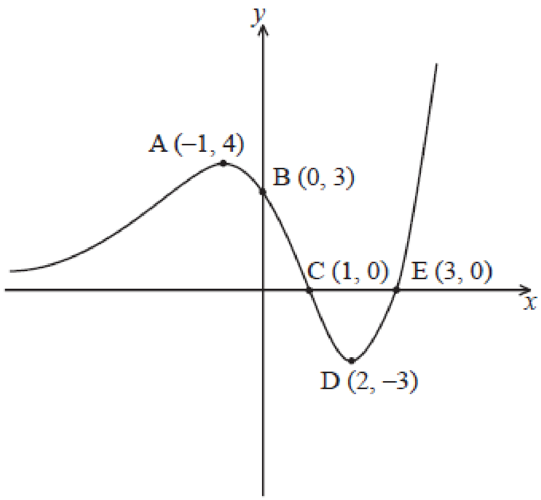
$\Rightarrow a = \frac{1+\sqrt{5}}{2}$ ***AG*** ***N0***

[8 marks]

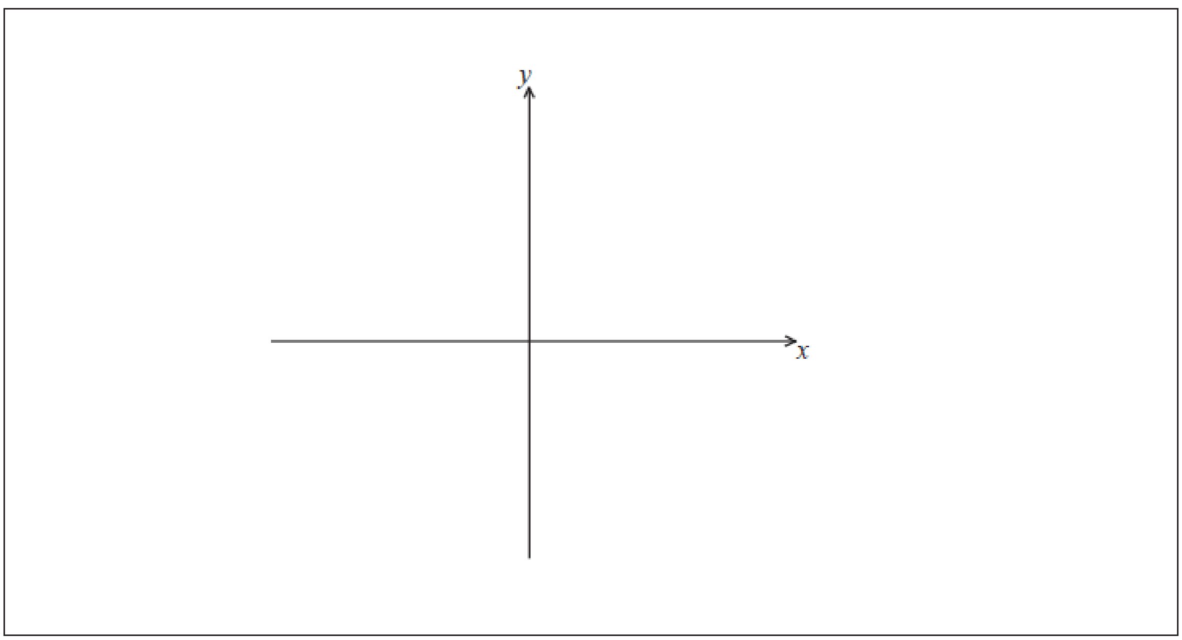
Examiners report

A fair amount of candidates had difficulties with this question. In part (a) many candidates were able to write down an expression for the volume in terms of a, but thereafter were largely unsuccessful. There is evidence that many candidates have lack of algebraic skills to manipulate the expression and obtain the volume in terms of x. In part (b) some candidates started with what they were trying to show to be true.

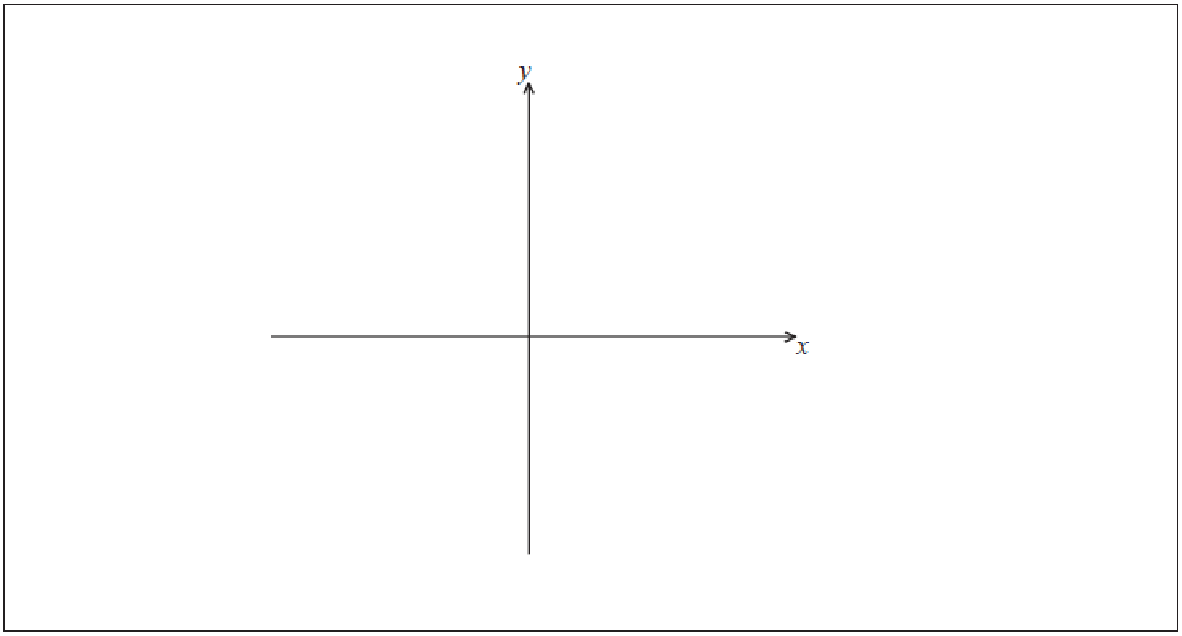
The graph of $y = f(x)$ is shown below, where A is a local maximum point and D is a local minimum point.



a. On the axes below, sketch the graph of $y = \frac{1}{f(x)}$, clearly showing the coordinates of the images of the points A, B and D, labelling them A', B', and D' respectively, and the equations of any vertical asymptotes. [3]

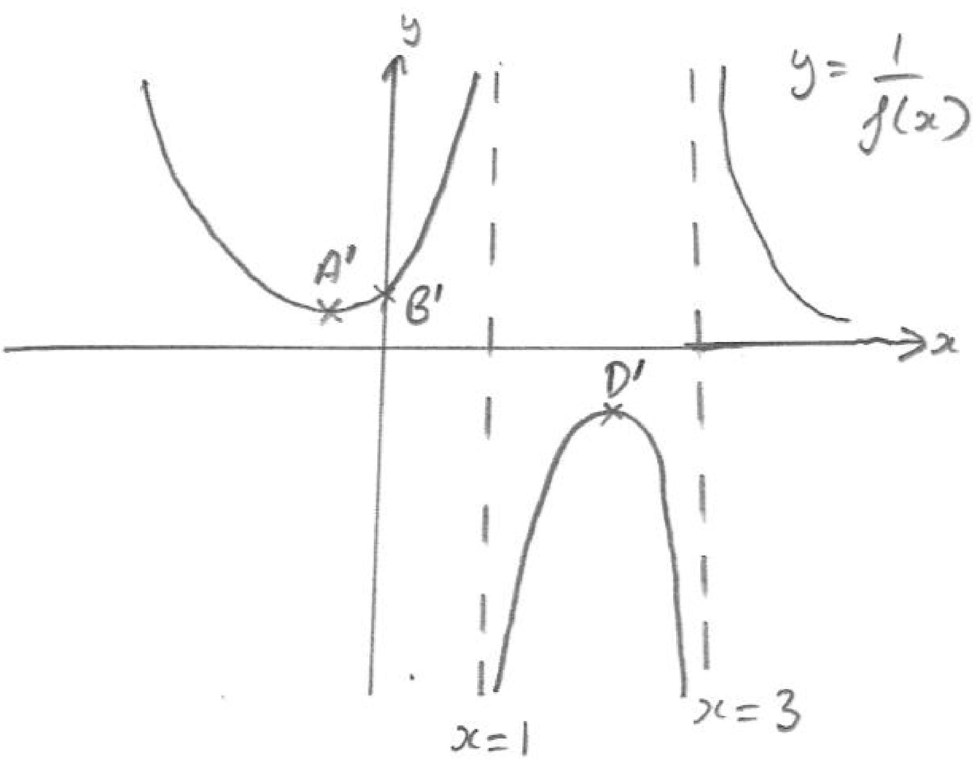


b. On the axes below, sketch the graph of the derivative $y = f'(x)$, clearly showing the coordinates of the images of the points A and D, labelling them A'' and D'' respectively. [3]



Markscheme

a.

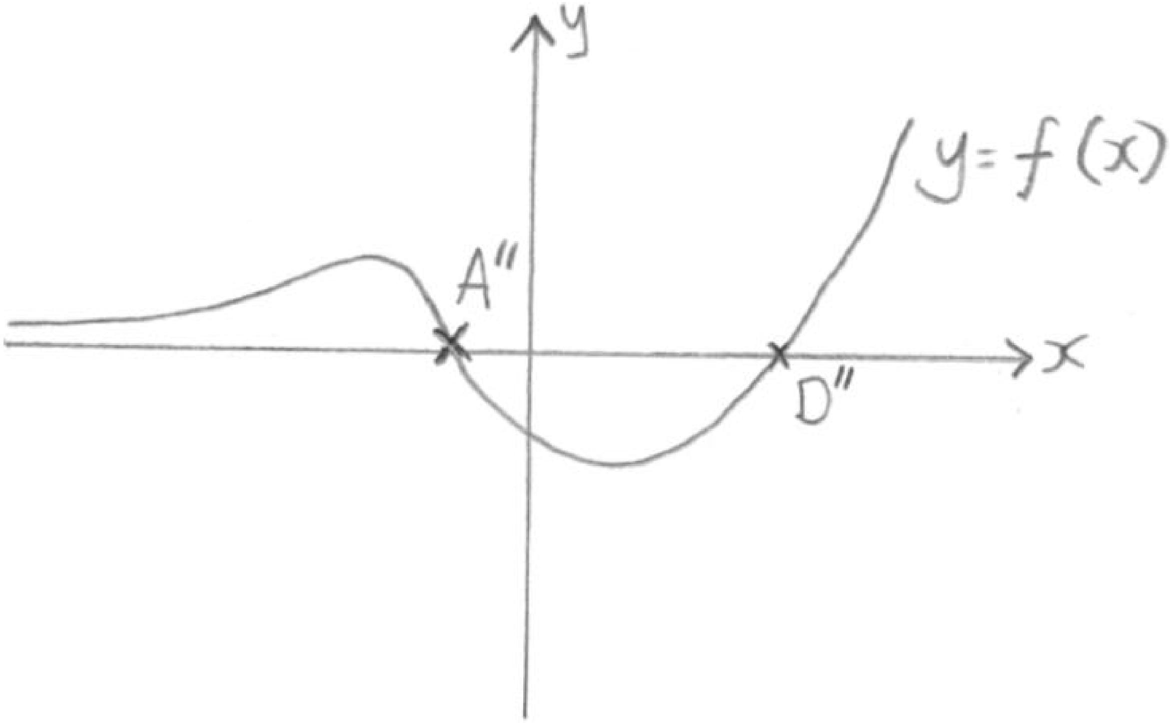


A1A1A1

Note: Award ***A1*** for correct shape.
Award ***A1*** for two correct asymptotes, and $x = 1$ and $x = 3$.
Award ***A1*** for correct coordinates, $\mathrm{A}'\left(-1, \frac{1}{4}\right)$, $\mathrm{B}'\left(0, \frac{1}{3}\right)$ and $\mathrm{D}'\left(2, -\frac{1}{3}\right)$.

[3 marks]

b.



A1A1A1

Note: Award ***A1*** for correct general shape including the horizontal asymptote.
Award ***A1*** for recognition of 1 maximum point and 1 minimum point.
Award ***A1*** for correct coordinates, $\mathrm{A}''(-1, 0)$ and $\mathrm{D}''(2, 0)$.

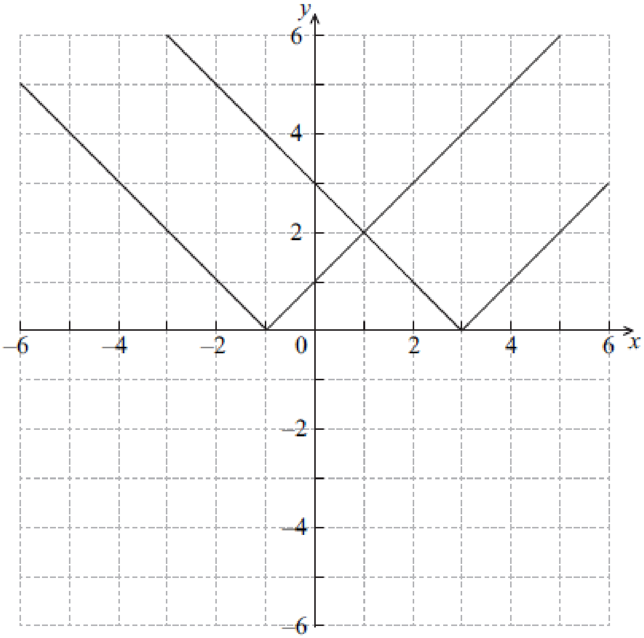
[3 marks]

Examiners report

a. Solutions to this question were generally disappointing. In (a), the shape of the graph was often incorrect and many candidates failed to give the equations of the asymptotes and the coordinates of the image points. In (b), many candidates produced incorrect graphs although the coordinates of the image points were often given correctly.

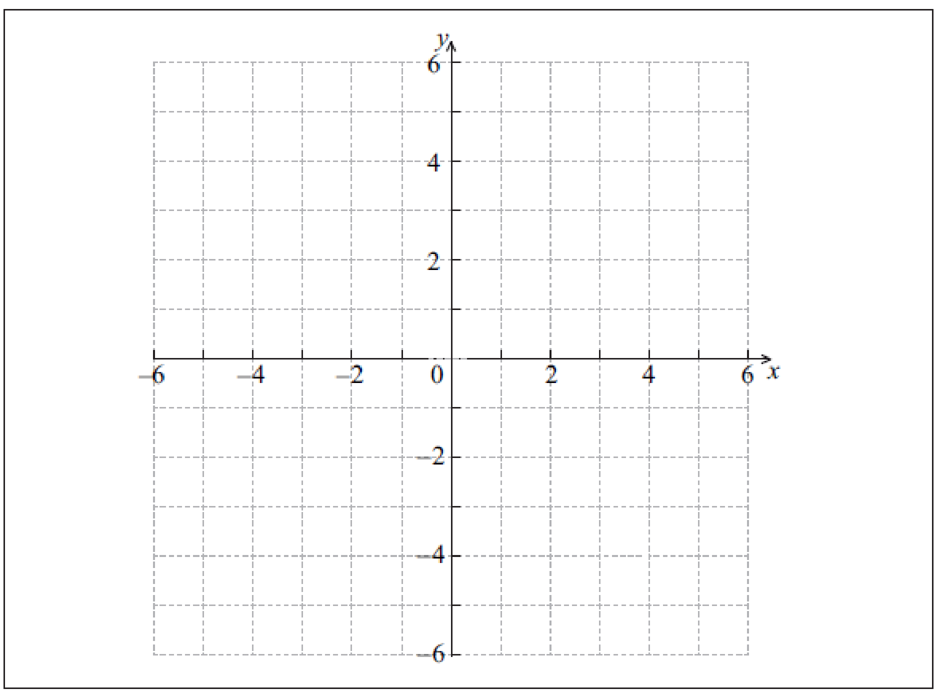
b. Solutions to this question were generally disappointing. In (a), the shape of the graph was often incorrect and many candidates failed to give the equations of the asymptotes and the coordinates of the image points. In (b), many candidates produced incorrect graphs although the coordinates of the image points were often given correctly.

The graphs of $y = |x + 1|$ and $y = |x - 3|$ are shown below.



Let $f(x) = |x+1| - |x-3|$.

a. Draw the graph of $y = f(x)$ on the blank grid below. [4]



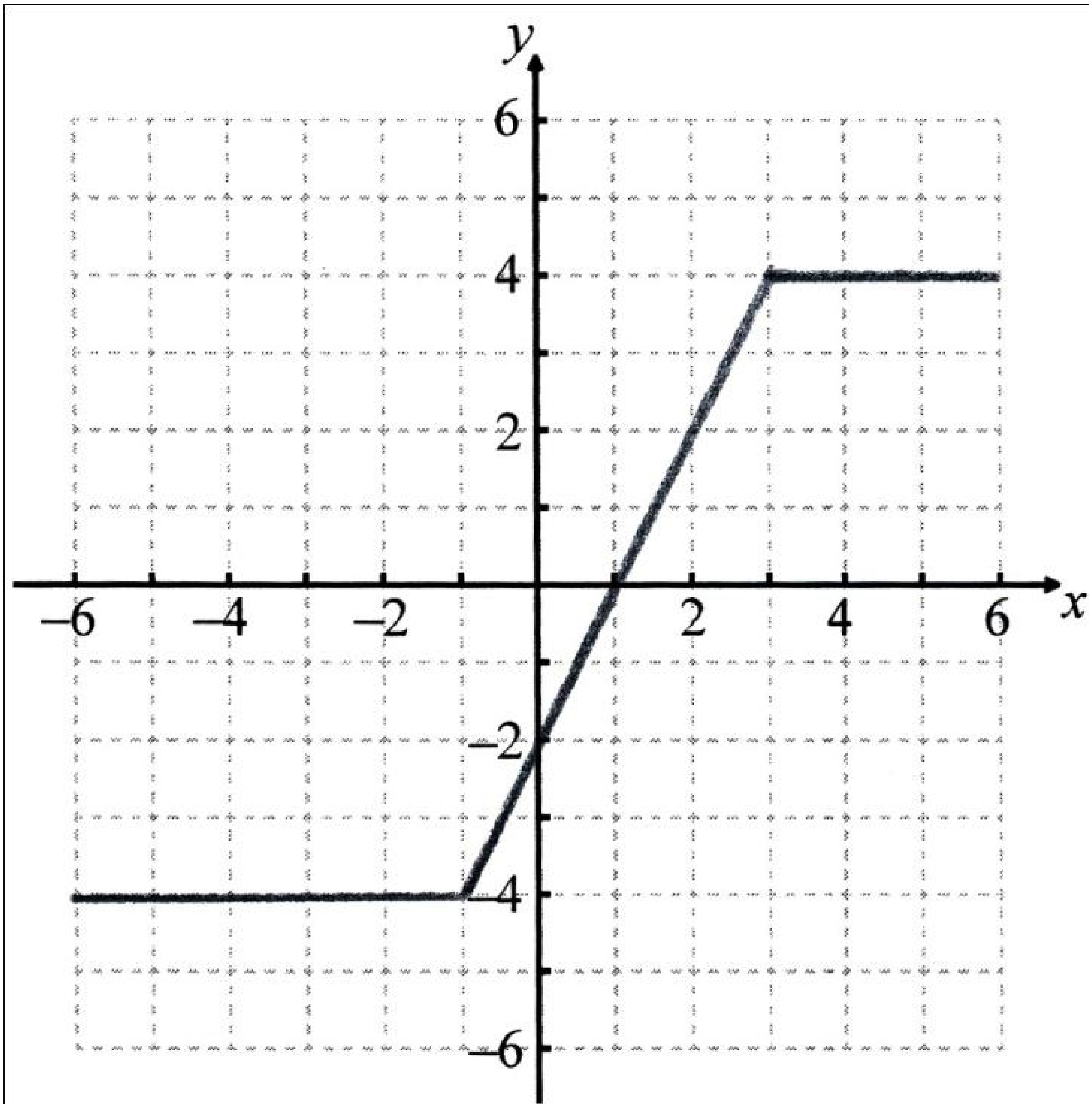
b. Hence state the value of [4]

(i) $f'(-3)$;

(ii) $f'(2.7)$;

(iii) $\int_{-3}^{-2} f(x)dx$.

Markscheme

a. 

M1A1A1A1

Note: Award ***M1*** for any of the three sections completely correct, ***A1*** for each correct segment of the graph. ***[4 marks]***

b. (i) 0 ***A1***

(ii) 2 ***A1***

(iii) finding area of rectangle ***(M1)***

−4 ***A1***

Note: Award ***M1A0*** for the answer 4.

[4 marks]

Examiners report

a. Most candidates were able to produce a good graph, and many were able to interpret that to get correct answers to part (b). The most common error was to give 4 as the answer to (b) (iii). Some candidates did not recognise that the "hence" in the question meant that they had to use their graph to obtain their answers to part (b).

b. Most candidates were able to produce a good graph, and many were able to interpret that to get correct answers to part (b). The most common error was to give 4 as the answer to (b) (iii). Some candidates did not recognise that the "hence" in the question meant that they had to use their graph to obtain their answers to part (b).